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OCEAN INTERNAL WAVE INDUCED MAGNETIC FIELDS WITHIN A SUBMERGED BUOY

I. W. Kay
W. Wasylkiwskyj

May 1980

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Ocean internal waves induce magnetic fields by virtue of motion of conducting sea water relative to the geomagnetic field. Measurements of such internal wave induced magnetic fields by sensitive magnetic instruments below the ocean surface require use of a protective enclosure or buoy. In this paper an analysis is presented of the effects of the enclosure on the magnetic fields and their spatial gradients when the enclosure (buoy) is spherical.

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CONTENTS

I. INTRODUCTION AND SUMMARY

The measurement of magnetic fields and magnetic field gradients in the ocean that are induced by the relative motion of sea water with respect to the geomagnetic field typically requires that the measuring instrument be enclosed in a measurement chamber or buoy. For example, when the instrument is a superconducting gradiometer (SQUID sensor) the minimal enclosure would be the cryogenic unit itself. The major contributors to the induced magnetic fields are believed to be ocean surface waves and internal waves, the latter providing the dominant contribution at frequencies below about 1 mHz for a moored sensor. Whereas formulas for computing the induced magnetic fields in the ocean are available [1], no analysis appears to have been carried out for estimating the effects of the enclosure on the detected magnetic field.

Basically, the buoy would introduce two types of disturbances: fluctuations of the magnetic field arising from the localized flow pattern dictated by the hydrodynamic properties of the buoy, and perturbations due to the sea water-air discontinuity at the boundary of the enclosure. The latter effect arises from the requirement that the normal component of electric current as well as the normal component of velocity vanish at the buoy surface.

The induced magnetic field within the buoy is determined essentially by an integral taken over the entire volume occupied by the velocity field exterior to the buoy. If, as would normally be the case, the volume occupied by the internal wave field is much larger than that occupied by the local flow around

the buoy, it is reasonable to suppose that the contribution to the induced magnetic field from the local flow velocity field may be neglected.

A more severe perturbation would be that due to the discontinuity at the buoy bounding surface. One readily convinces himself of the importance of this effect if one recalls that the asymmetry of the gradients of the magnetic field in the water is removed once the measurement is carried out in air (i.e., within the buoy): rotating the axis of a hypothetical SQUID sensor in air by 90 deg would produce no change in the measured gradient, whereas different results would be obtained if such a rotation were carried out in sea water. This is simply a consequence of the fact that the curl of the induced magnetic field vanishes identically in air, but not in sea water, where there exist localized electric currents.

In this paper an analysis is presented of the effects of the discontinuity at the buoy walls on the magnetic fields and gradients induced by internal waves in the ocean. An exact solution for the fields within a buoy of arbitrary shape appears rather difficult. Here, for simplicity, the buoy is modeled as a sphere. Although this shape hardly constitutes a practical contour for a submerged instrument package, it does provide a fairly tractable model for estimating the general trend of the boundary effects. In addition, the methodology can provide guidance for future analyses of more realistic shapes. Even in the case of the sphere, an exact solution is not trivial. The approximations underlying the present analysis are that the sphere radius is small by comparison with the spatial wavelength of the unperturbed internal wave field and by comparison with the depth of the buoy below the ocean surface.

The first of these two approximations is equivalent to assuming that the unperturbed internal wave field is essentially constant over the volume of sea water displaced by the buoy.

For spatial wavelengths of principal interest herein (1,000 to 100 meters), this is certainly a reasonable assumption.

The technique of solution involves a perturbation expansion in ascending powers of the sphere radius. When the sphere radius is small in relation to the spatial period of the hydrodynamic flow field, only the first term of the perturbation expansion needs to be retained. For notational convenience, in the analytical development only the radius of the sphere a, rather than the dimensionless ratio a/λ is referred to explicitly as the small expansion in parameter. (See discussion on p. 44 following Eq. (122)).

In Section II the general mathematical framework is set forth. The rationale for the perturbation technique is developed in Section III and Appendix A. In Section IV an expression for the magnetic field is obtained which is valid in the zeroth order approximation. The final result is embodied in Eq. (72), which appears to bear a strong resemblance to the well-known "cavity" definition of electromagnetic field quantities in an extended medium. A formula for the "correction" term involving the internal wave parameters explicitly is given by Eq. (83), while the corresponding unperturbed field components (i.e., in the absence of the buoy) are given by Eq. (78). These results show that the perturbation of the field due to the buoy boundary is of the same order of magnitude as the original field in the ocean.

In the zeroth order approximation the field is constant within the buoy. Thus, in order to estimate the spatial gradients, the next higher-order term must be retained in the perturbation expansion. The analysis is carried out in Section V. The final result is given by Eq. (122). In Section VI this formula is employed in the computation of magnetic field gradients, the general formula for which is Eq. (124). Despite its complexity, the result admits of a simple physical interpreta-

tion: the effect of the enclosure is to symmetrize the unperturbed gradient and to add a rotational shear term. The symmetrization is in the form $1/2[G_{pq}^{(P)} + G_{qp}^{(P)}]$, where $G_{pq}^{(P)}$ is the gradient of the q-th component of the magnetic field in the direction p, in the absence of the buoy. As noted earlier, $G_{pq}^{(P)} \neq G_{qp}^{(P)}$ in sea water, whereas in the buoy the gradients must be symmetric in the indices by virtue of the vanishing of the curl of the magnetic field in air. The net effect of the shear term is harder to interpret. In some numerical calculations performed thus far it appears to be small.

Equation (135) translates (124) into a formula in which the dependence on the internal wave parameters is made explicit. Unlike in the case of the magnetic field, Eqs. (78) and (83), the formula for the gradients cannot be interpreted as a sum of an unperturbed quantity plus a perturbation term. Equation (135) is used in Section VII to compute the correlation function functions and spectra for ocean internal waves. Both moored and towed situations are considered. In the latter case the fast tow approximation is employed in conjunction with the hypothesis of Milder as given in [1]. The final formulas, although quite cumbersome in appearance, reduce the computation to a series of quadratures involving the Väisälä frequency. The detailed sequence of steps required to implement the spectral calculations on a computer are presented in Appendix C.

Only a few numerical calculations using the theory here developed have thus far been carried out. Figure 1 shows some preliminary results for a unidirectional single frequency internal wave. The Väisälä frequency profile is assumed exponentially decreasing with the maximum Väisälä frequency of .833 mHz at the ocean surface and a decay constant of 1300 m. A first mode internal wave of 1 meter maximum amplitude and a wavelength of 200 meters (frequency .716 mHz) is assumed. The amplitude of the internal wave displacement as a function of depth is indicated by the broken curve (peak at about 100 m depth).

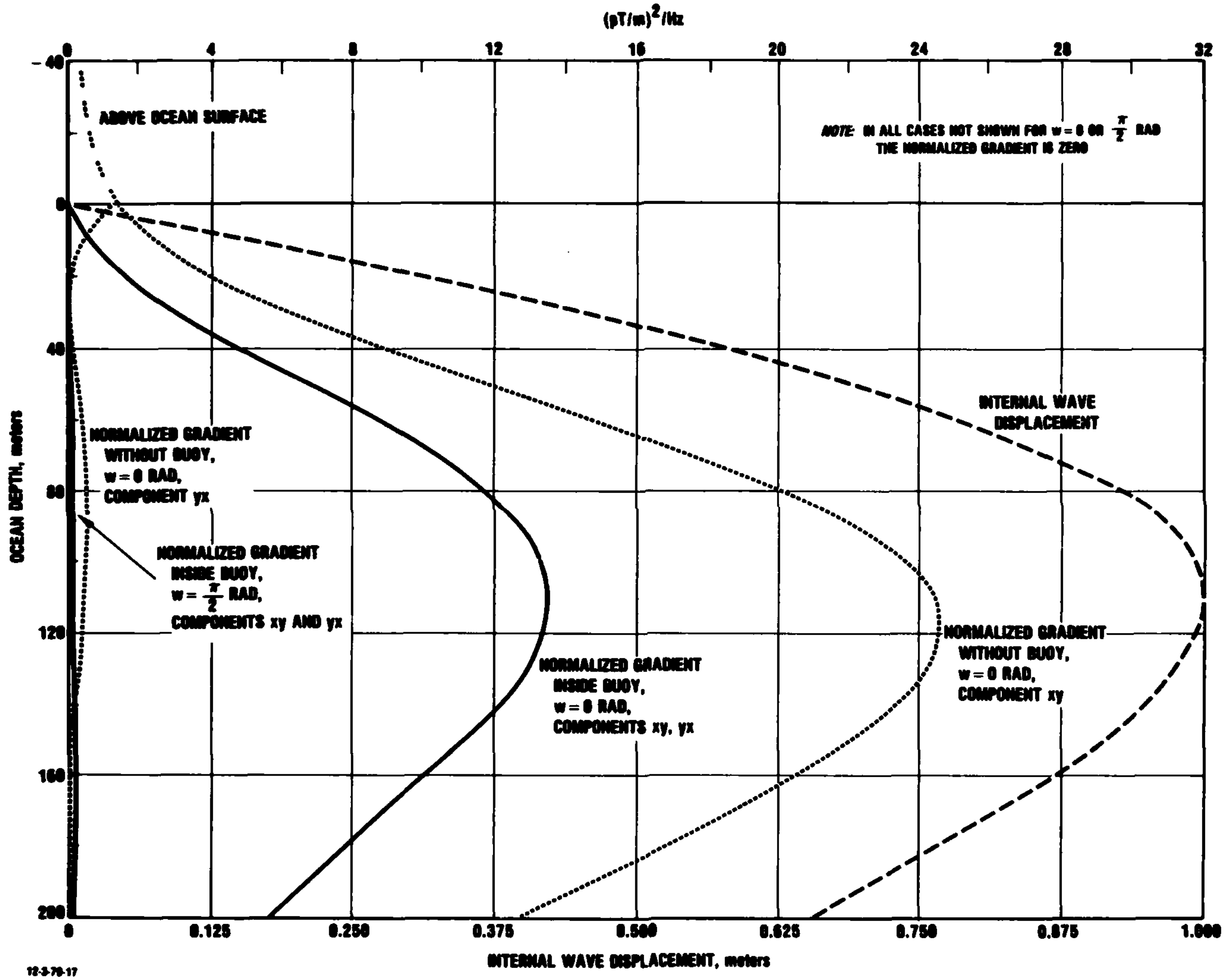

FIGURE 1. Magnetic Field Gradient in an Exponentially Stratified Ocean (Internal Wave Has 1.0-Meter Displacement at Mode Peak).

The spectral peak of the gradient is plotted for an integration time of 1000 sec (Väisälä period ≈ 1200 sec). The geomagnetic field is assumed purely horizontal (equatorial zone) and along the x direction (Fig. 2). The direction of travel of the internal wave relative to the geomagnetic field is denoted by w in the figure. (w = 0, wave direction along the geomagnetic field, w = $\pi/2$, wave direction normal to the geomagnetic field). The gradients are defined as follows: component xy is the derivative of the y (vertical) component of the field along the x direction. (The coordinate system is shown in Fig. 2.)

II. FORMULATION OF THE PROBLEM

The sea surface is assumed to be plane and sea water occupies the region $y < D$, shown in Fig. 2. At a depth D we assume a spherical enclosure of radius a, which, to the first approximation, constitutes our model for a buoy containing the magnetic sensing instrument (e.g., a superconducting gradiometer). For the purpose of the following analysis the interior of the buoy is filled with air. Our objective is to obtain, in the interior of the spherical enclosure, an approximation to the magnetic field and its spatial gradients that are induced by the interaction of ocean internal waves with the geomagnetic field. Analytical results for computing the induced magnetic field in the ocean in the absence of an enclosure are presented in [1]. The analysis presented herein relies heavily on the results and notation employed in [1]. As in [1], we shall be interested only in internal waves with periods much shorter than the inertial period. Consequently, the vorticity vector $\underset{\sim}{\omega}(\underset{\sim}{r},t)$ can be assumed parallel to the ocean surface. We denote by $\underset{\sim}{V}(\underset{\sim}{r},t)$ the fluid velocity so that

$$\underset{\sim}{\omega}(\underset{\sim}{r},t) = \nabla \times \underset{\sim}{V}(\underset{\sim}{r},t)\,. \tag{1}$$

We make the usual incompressibility assumption and express the fluid velocity in terms of the vector stream function $\underset{\sim}{\psi}(\underset{\sim}{r},t)$, viz.,

$$\underset{\sim}{V}(\underset{\sim}{r},t) = \nabla \times \underset{\sim}{\psi}(\underset{\sim}{r},t). \tag{2}$$

Unless germain to the discussion, we shall henceforth omit the explicit dependence on time in the arguments of the velocity, the vorticity and the stream function.

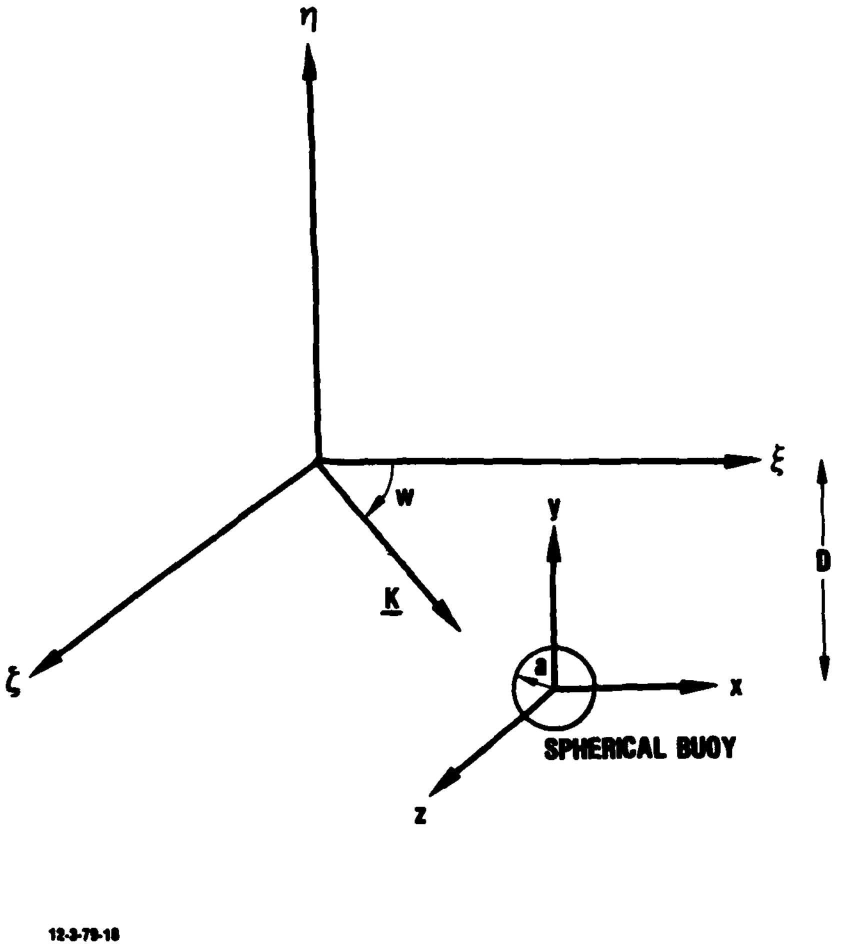

FIGURE 2. Coordinate System.

In the course of the analysis we shall use Green's functions of the Laplace equation pertaining to the regions within and outside the sphere. We adopt the following notation. The symbols S and P will be employed to denote the surface of the sphere and ocean, respectively. The region inside the sphere is designated by V_S. The water region between the surfaces S and P is designated by V_{PS} and the entire undersea region below the ocean surface by V_P. Volume integrals are indicated by the integration region V_S, V_{PS} or V_P and by the volume element by dv. Surface integrals are indicated, similarly, by the region of integration S or P and by the surface element ds. The surface of the unit sphere is labeled Ω and the corresponding surface element $d\Omega$.

The free space Green's function is designated by G_o; i.e.,

$$G_o(\underset{\sim}{r}, \underset{\sim}{r}') = \frac{1}{4\pi|\underset{\sim}{r}-\underset{\sim}{r}'|} . \tag{3}$$

The Dirichlet Green's function that vanishes on both the surfaces P and S is designated by G_D. The Dirichlet Green's functions that vanish on P or on S alone are designated by G_{DP} or G_{DS}. Similar designations, with N replacing D, are used for the Neumann Green's function whose normal derivatives vanish on the specified surface or surfaces.

The vertical displacement of the ocean surface plays a very minor role in the internal wave motion in the ocean. Consequently, the boundary condition at the ocean surface will be taken as $V_y = 0$. Also, the buoy will be assumed rigid so that the normal component of fluid velocity on the surface of the sphere must be zero.

We first review briefly the formulation for the induced magnetic field in the ocean in the absence of a buoy. The geomagnetic field is assumed constant and is denoted by $\underset{\sim}{B}_o$. As shown in [1], the electric current $\underset{\sim}{J}$, induced by the motion of sea water, is given by

$$\underset{\sim}{J}_P = \sigma\,[-\nabla\phi' + (\underset{\sim}{B}_o \cdot \nabla)\underset{\sim}{\psi}_P], \tag{4}$$

and the electrostatic potential in the water is expressed by

$$\phi = \phi' - \underset{\sim}{B}_o \cdot \underset{\sim}{\psi}_P . \tag{5}$$

The scalar potential function ϕ' satisfies

$$\nabla^2\phi' = 0. \tag{6}$$

The stream function $\underset{\sim}{\psi}_P$ is obtained from the vorticity $\underset{\sim}{\omega}$, which we regard as a prescribed source function. Since

$$\underset{\sim}{\omega} = \nabla \times \nabla \times \underset{\sim}{\psi}_P = \nabla\nabla\cdot\underset{\sim}{\psi}_P - \nabla^2\underset{\sim}{\psi}_P, \tag{7}$$

we may use the gauge condition

$$\nabla \cdot \underset{\sim}{\psi}_P = 0 \tag{8}$$

to obtain

$$\nabla^2 \underset{\sim}{\psi}_P = - \underset{\sim}{\omega} \, . \tag{9}$$

We solve (9) subject to the boundary condition on $\underset{\sim}{\psi}_P$ that ensures $V_y = 0$ at the ocean surface and is, at the same time, compatible with the gauge condition expressed by (8). The boundary condition on the scalar potential ϕ' in (6) is obtained from the requirement that the normal component of electric current (4) vanishes at the ocean-air interface. When the vorticity $\underset{\sim}{\omega}$ has no vertical component, $\underset{\sim}{\psi}_P$ is also purely horizontal and one finds that

$$\frac{\partial \phi'}{\partial y} = 0 \text{ on } P \, . \tag{10}$$

One then concludes that $\nabla\phi' \equiv 0$ everywhere and the electric current becomes

$$\underset{\sim}{J}_P = \sigma(\underset{\sim}{B}_o \cdot \nabla)\underset{\sim}{\psi}_P \tag{11}$$

In this special case the boundary condition $\underset{\sim}{\psi}_P = 0$ at P ensures that $V_y = 0$ at P. The solution of (9) is then given by

$$\underset{\sim}{\psi}_P = \int_{V_P} G_{DP} \, \underset{\sim}{\omega} \, dv \, . \tag{12}$$

One can show directly from (12) that the gauge condition (8) is satisfied. This ensures that the vorticity $\underset{\sim}{\omega} = \nabla x \nabla x \underset{\sim}{\psi}_P$ with $\underset{\sim}{\psi}_P$ as computed from (12) is identical with that prescribed in the integrand. The induced magnetic field $\underset{\sim}{B}_P$ is determined with the aid of the vector potential $\underset{\sim}{A}_P$, which satisfies

$$\nabla^2 \underset{\sim}{A}_P = -\mu_o \, \underset{\sim}{J}_P, \tag{13}$$

subject to the gauge condition

$$\nabla \cdot \underset{\sim}{A}_P = 0. \tag{14}$$

The solution of (13) is

$$\underset{\sim}{A}_P = \int_{V_P} \mu_o \, G_o \underset{\sim}{J}_P dv \; , \tag{15}$$

and the magnetic field is found from the formula

$$\underset{\sim}{B}_P = \nabla \times \underset{\sim}{A}_P \; . \tag{16}$$

We now consider the problem of finding the magnetic field in the spherical enclosure. We have

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} \tag{17}$$

with

$$\underset{\sim}{A} = \int_{V_{PS}} \mu_o \, G_o \, \underset{\sim}{J} \, dv, \tag{18}$$

where the current $\underset{\sim}{J}$ is to be determined from

$$\underset{\sim}{J} = \sigma[-\nabla\phi' + (\underset{\sim}{B}_o \cdot \nabla)\underset{\sim}{\psi}]. \tag{19}$$

The scalar potential ϕ' is no longer zero, since in addition to the boundary condition (10), one must have

$$-\frac{\partial\phi'}{\partial r} + \underset{\sim}{i}_r \cdot (\underset{\sim}{B}_o \cdot \nabla)\underset{\sim}{\psi} = 0. \tag{20}$$

on the surface of the sphere. (Here $\underset{\sim}{i}_r$ is the radial unit vector). We again consider only a purely horizontal vorticity function. Consequently, at the ocean surface

$$\underset{\sim}{\psi} = 0 \text{ at P.} \tag{21}$$

On the spherical surface we require $\underset{\sim}{i}_r \cdot \underset{\sim}{V} = 0$, or, equivalently,

$$\underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{\psi} = 0 \quad \text{on S.} \tag{22}$$

We prescribe the vorticity function $\underset{\sim}{\omega}$ in V_{PS} and determine $\underset{\sim}{\psi}$ from

$$\nabla \times \nabla \times \underset{\sim}{\psi} = \underset{\sim}{\omega}, \tag{23}$$

subject to boundary conditions (21) and (22). An exact solution of this problem is rather difficult. We shall be interested only in a small spherical enclosure in which case one may employ the following approximate solution:

$$\underset{\sim}{\psi} = \int_{V_{PS}} G_D \, \underset{\sim}{\omega} \, dv \ . \tag{24}$$

Equation (24) gives a stream function which vanishes not only at the ocean surface but also on the surface of the sphere. Although the latter condition is compatible with (22), the *divergence of* $\underset{\sim}{\psi}$ turns out not to be equal to zero. Consequently, the vorticity as computed with the aid of (24) does not agree with the vorticity originally prescribed in the problem. Instead, one obtains

$$\underset{\sim}{\omega}_{actual} = \nabla \times \nabla \times \int_{V_{PS}} G_D \, \underset{\sim}{\omega} \, dv$$

$$= (\nabla\nabla\cdot - \nabla^2) \int_{V_{PS}} G_D \underset{\sim}{\omega} \, dv = \underset{\sim}{\omega} + \nabla\nabla\cdot \int_{V_{PS}} G_D \, \underset{\sim}{\omega} \, dv \ . \tag{25}$$

As the radius of the sphere tends to zero, $G_D \to G_{DP}$, so that the last number in (25) must be O(a). Consequently, for a small* sphere, we have

$$\underset{\sim}{\omega}_{actual} \approx \underset{\sim}{\omega} \ . \tag{26}$$

*I.e., small in the sense that the velocity field (e.g., the stream function) is constant over the sphere volume.

We shall be interested in computing the magnetic field and its spatial gradients inside the spherical buoy when the radius a is small. In this case it is more convenient to deal with the perturbed quantities $\underset{\sim}{B} - \underset{\sim}{B}_P$, $\underset{\sim}{A} - \underset{\sim}{A}_P$, $\underset{\sim}{\psi} - \underset{\sim}{\psi}_P$. The first two are related by

$$\underset{\sim}{B} - \underset{\sim}{B}_P = \nabla \times (\underset{\sim}{A} - \underset{\sim}{A}_P). \tag{27}$$

Because of (15) and (18)

$$\underset{\sim}{A} - \underset{\sim}{A}_P = \mu_o \int_{V_{PS}} (\underset{\sim}{J} - \underset{\sim}{J}_P) G_o dv - \mu_o \int_{V_S} \underset{\sim}{J}_P G_o dv, \tag{28}$$

where the observation point $\underset{\sim}{r}$ of the Green's function is inside the sphere, i.e., $|\underset{\sim}{r}| = r < a$.

III. PERTURBATION SOLUTION UNDER THE ASSUMPTION OF A SMALL SPHERE RADIUS

The free space Green's function G_o given by (3) has the power series expansion, when $r < r'$,

$$G_o(\underset{\sim}{r},\underset{\sim}{r}') = \frac{1}{4\pi r'} \sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n P_n(\cos\gamma), \tag{29}$$

where P_n is the Legendre polynomial of order n and γ is the angle between the vectors $\underset{\sim}{r}$ and $\underset{\sim}{r}'$. It follows that for the first term on the right in (28)

$$\mu_o \int_{V_{PS}} (\underset{\sim}{J}-\underset{\sim}{J}_P)\, G_o\, dv' = \sum_{n=0}^{\infty} \underset{\sim}{\alpha}_n\, r^n, \tag{30}$$

where

$$\underset{\sim}{\alpha}_n = \frac{\mu_o}{4\pi} \int_{V_{PS}} \frac{\underset{\sim}{J}-\underset{\sim}{J}_P}{r'^{\,n+1}}\, P_n(\cos\gamma)\, dv' . \tag{31}$$

In (30) and (31) dv' has been written in place of dv to emphasize that integration is with respect to primed variables associated with G_o and that the result of the integration is still a function of the unprimed variables. Since $P_o = 1$ it should be noted that $\underset{\sim}{\alpha}_o$ = constant. It follows that

$$\mu_o \nabla \times \int_{V_{PS}} (\underset{\sim}{J}-\underset{\sim}{J}_P)\, G_o\, dv' = \sum_{n=1}^{\infty} \left(\nabla \times \underset{\sim}{\alpha}_n + \frac{n\underset{\sim}{i}_r \times \underset{\sim}{\alpha}_n}{r}\right) r^n. \tag{32}$$

In accounting for the second term on the right of (28) the identity

$$\int_{V_S} \nabla' \times (G_o \underset{\sim}{J}_P)\, dv' = \int_S \underset{\sim}{i}_r' \times \underset{\sim}{J}_P G_o\, ds' \tag{33}$$

implies that

$$\nabla \times \int_{V_S} \underset{\sim}{J}_P \, G_o dv' = \int_{V_S} \nabla' \times \underset{\sim}{J}_P G_o dv' - \int_S \underset{\sim}{i}'_r \times \underset{\sim}{J}_P G_o ds'. \tag{34}$$

Then, using the power series expansion (29) for G_o when $r < r'$ and the expansion obtained by interchanging r and r' when $r > r'$, it follows that

$$\nabla \times \int_{V_S} \underset{\sim}{J}_P \, G_o dv' = \frac{1}{4\pi} \sum_{n=0}^{\infty} r^n \int_{\Omega} \int_{r}^{a} \frac{\nabla' \times \underset{\sim}{J}_P}{r'^{n-1}} P_n (\cos\gamma) dr' d\Omega$$

$$+ \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_{\Omega} \int_{0}^{r} \nabla' \times \underset{\sim}{J}_P P_n (\cos\gamma) r'^{n+2} dr' d\Omega$$

$$- \frac{a}{4\pi} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \int_{\Omega} (\underset{\sim}{i}_r \times \underset{\sim}{J}_P)\big|_{r'=a} P_n(\cos\gamma) d\Omega$$

$$= \frac{-a}{4\pi} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \int_{\Omega} \underset{\sim}{i}_{r'} \times \underset{\sim}{J}_P(0) P_n(\cos\gamma) d\Omega + O(a^2), \tag{35}$$

where $O(a^2)$ has been written in place of functions of the form $a^2 \sum_{n=0}^{\infty} c_n \left(\frac{r}{a}\right)^n$. In deriving (34) use has been made of the continuity of $\underset{\sim}{J}_P$ to relate $\underset{\sim}{J}_P$ on S to its value at the center of the sphere, i.e.,

$$\underset{\sim}{J}_P\big|_{r'=a} = \underset{\sim}{J}_P(0) + O(a).$$

It follows from (27), (28), (32), and (35) that

$$\underset{\sim}{i}_r \cdot (\underset{\sim}{B} - \underset{\sim}{B}_P) = \sum_{n=0}^{\infty} (\underset{\sim}{i}_r \cdot \nabla \times \alpha_n) r^n$$

$$+ \mu_o \frac{a}{4\pi} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \int_{\Omega} \underset{\sim}{i}_r \cdot \underset{\sim}{i}_{r'} \times \underset{\sim}{J}_P(0) P_n(\cos\gamma) d\Omega + O(a^2) .$$

$$= \sum_{n=1}^{\infty} (\underset{\sim}{i}_r \cdot \nabla x \alpha_n) r^n + \frac{-\mu_o a}{4\pi} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \underset{\sim}{J}_P(0) \cdot \int_{\Omega} \underset{\sim}{i}_r x \underset{\sim}{i}_{r'} P_n(\cos\gamma) d\Omega$$

$$+ 0\ (a^2)$$

$$= \sum_{n=1}^{\infty} (\underset{\sim}{i}_r \cdot \nabla x \underset{\sim}{\alpha}_n) r^n + 0\ (a^2). \tag{36}$$

The last relation follows from the fact that the integral in the penultimate relation satisfies

$$\int_{\Omega} \underset{\sim}{i}_r \times \underset{\sim}{i}_{r'}\ P_n(\cos\gamma) d\Omega = 0. \tag{37}$$

This can be seen from the fact that, if $\underset{\sim}{i}_r$ is taken to be the polar axis relative to the primed coordinates in (37), i.e., by rotating coordinates in the integrand, then

$\underset{\sim}{i}_r \times \underset{\sim}{i}_r' = \underset{\sim}{i}_{\phi}$, is the azimuthal unit vector

and

$P_n(\cos\gamma) = P_n(\cos\theta)$. Since $d\Omega = \sin\theta d\theta d\phi$ one readily verifies that integration over a complete sphere yields zero.

According to (36), on the spherical surface,

$$\underset{\sim}{i}_r \cdot \underset{\sim}{B}|_{r=a} = \underset{\sim}{i}_r \cdot \underset{\sim}{B}_P|_{r=a} + \sum_{n=1}^{\infty} (\underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{\alpha}_n)_{r=a} a^n + 0(a^2)$$

$$= b_a(\theta,\phi) + 0(a^2). \tag{38}$$

Since inside the sphere

$$\nabla \times \underset{\sim}{B} = 0,$$

there exists a scalar function Φ in terms of which

$$\underset{\sim}{B} = \nabla\ \Phi \tag{39}$$

inside the sphere. Also

$$\nabla \cdot \underset{\sim}{B} = \nabla^2 \Phi = o \tag{40}$$

inside the sphere, and (38) and (39) imply that on the surface of the sphere

$$\frac{\partial \Phi}{\partial r}\Big|_{r=a} = b_a(\phi,\theta) + 0\ (a^2). \tag{41}$$

To find the magnetic field within the spherical enclosure, it is only necessary to solve Laplace's equation for Φ in V_S subject to the boundary condition (41) on S. The solution for small a does not vanish in the limit as a approaches zero, but satisfies the Laplace equation boundary value problem with the radial derivative of Φ equal to $b_o(\theta,\phi)$, a quantity given by

$$b_o(\theta,\phi) = \lim_{a \to 0} \left[\underset{\sim}{i}_r \cdot \underset{\sim}{B}_P \Big|_{r=a} + \sum_{n=1}^{\infty} (\underset{\sim}{i}_r \cdot \nabla x \underset{\sim}{\alpha}_n) \Big|_{r=a} a^n \right]. \tag{42}$$

In calculating the magnetic field inside the sphere for small a, it is appropriate to use a power series in r, or rather, $\frac{r}{a}$. Accordingly, the potential function Φ is

$$\Phi = r \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^{n-1} Z_n(\phi,\theta), \tag{43}$$

where the $Z_n(\phi,\theta)$ are the terms in the spherical harmonic expansion* of the boundary value:

*The $Z_n(\phi,\theta)$ denotes the sum $\sum_{m=-n}^{n} d_m P_n^m(\cos\theta) e^{im\phi}$, where d_m are suitable constants and $P_n^m(\cos\theta)$ are associated Legendre polynomials.

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = \sum_{n=1}^{\infty} Z_n(\phi,\theta) \approx b_a(\phi,\theta) \quad . \tag{44}$$

The fact that the series in (43) and (44) are both missing the term corresponding to n=0 is a consequence of Gauss' theorem and the fact that the divergence of any magnetic field is zero. That is, the integral over S of the magnetic field's normal component vanishes, but the same integral provides the coefficient of the zeroth order term in the normal component's spherical harmonic expansion.

According to (43),

$$\underset{\sim}{B} = \nabla\,\Phi = \sum_{n=1}^{\infty} \left[Z_n(\phi,\theta)\underset{\sim}{i}_r + \frac{1}{n}\,\nabla_\Omega\, Z_n\right] \left(\frac{r}{a}\right)^{n-1} \quad . \tag{45}$$

In (45) the operator ∇_Ω is the angular part of the gradient and is defined by

$$\nabla_\Omega = \underset{\sim}{i}_\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\,\underset{\sim}{i}_\phi\,\frac{\partial}{\partial\phi} \quad .$$

IV. EXPLICIT EVALUATION OF THE MAGNETIC FIELD TO THE ZEROTH ORDER IN a

The coefficients in the power series (45) for the magnetic field are obtained from terms in the spherical harmonic expansion (44) of the function $b_a(\theta,\phi)$, given by

$$b_a(\phi,\theta) = \underset{\sim}{i}_r \cdot \underset{\sim}{B}_P\big|_{r=a} + \sum_{n=1}^{\infty} (\underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{\alpha}_n)_{r=a} a^n. \tag{46}$$

The first term on the right of (46) is presumably known, and is, in fact, to the lowest order in a, the radial component of the unperturbed magnetic field at the center of the sphere. The remaining terms must be obtained from the vectors $\underset{\sim}{\alpha}_n$ defined by (31).

According to (19) and (11), the current appearing in (31) is given by

$$\underset{\sim}{J} - \underset{\sim}{J}_P = \underset{\sim}{J}_M + \underset{\sim}{J}_E, \tag{47}$$

where

$$\underset{\sim}{J}_M = \sigma\ (\underset{\sim}{B}_o \cdot \nabla)\ (\underset{\sim}{\psi} - \underset{\sim}{\psi}_P)$$

and

$$\underset{\sim}{J}_E = -\sigma\ \nabla\ \phi'.$$

It will now be shown that the term $\underset{\sim}{J}_E$ contributes a term of order a to the magnetic field inside the spherical enclosure. It will be found later that the contribution of $\underset{\sim}{J}_M$ is of order one in a.

Because of (27), (28) and (29) the contribution of $\underset{\sim}{J}_E$ to the magnetic field $\underset{\sim}{B}_E$ is

$$\underset{\sim}{B}_E = -\mu_o\sigma \, \nabla \, x \int_{V_{PS}} \nabla\phi' \; G_o dv' = -\mu_o\sigma \int_{V_{PS}} \nabla G_o x \nabla\phi' dv'$$

$$= \mu_o\sigma \int_{V_{PS}} \nabla' G_o x \nabla\phi' dv' = \mu_o\sigma \int_{V_{PS}} \nabla' x (G_o \nabla\phi') dv' \tag{48}$$

$$= \mu_o\sigma \int_{P+S} n' x \nabla\phi' G_o ds'.$$

The function ϕ' in (48) is the solution of the boundary value problem given by (6), (10), and (20) in which the normal derivative of ϕ' is given on S and P. It is shown in Appendix A that the solution of a similar problem, in which the function rather than its normal derivative is prescribed on S and P can be obtained approximately by ignoring the boundary condition on P. The approximate solution is correct to the lowest order in the ratio $\frac{a}{D}$. A similar argument can be given for the case in which the normal derivative is prescribed rather than the function. Accordingly, it can be asserted that the boundary value problem determined by (6) and (20) gives ϕ' correctly except for terms of order $\frac{a}{D}$.

Thus, ϕ' can be written in the form

$$\phi' = -\sum_{n=0}^{\infty} \frac{a^{n+2}}{(n+1)r^{n+1}} Z_n(\phi,\theta) + 0(\tfrac{a}{D}), \tag{49}$$

where the $Z_n(\phi,\theta)$ are the terms in the spherical harmonic expansion of the boundary value on S:

$$\frac{\partial\phi'}{\partial r}\Big|_{r=a} = \underset{\sim}{i}_r \cdot (\underset{\sim}{B}_o\cdot \nabla)\underset{\sim}{\psi}\Big|_{r=a}.$$

Then $\nabla\phi'$ has the form

$$\nabla\phi' \sim \sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n+2} \underset{\sim}{\zeta}_n(\phi,\theta) + 0\left(\frac{a}{D}\right), \qquad (50)$$

where

$$\underset{\sim}{\zeta}_n(\phi,\theta) = Z_n(\phi,\theta)\,\underset{\sim}{i}_r - \frac{1}{n+1}\nabla_\Omega Z_n(\phi,\theta).$$

Neglecting terms of order $\frac{a}{D}$ in (50) is equivalent to dropping the integral over P in (48). Then

$$\underset{\sim}{B}_E \sim \mu_o\sigma\int_S \underset{\sim}{i}_{r'} \times \nabla\phi' G_o ds'. \qquad (51)$$

From (50), (51), and the power series expansion (29) of G_o for an observation point inside the sphere it follows that

$$\underset{\sim}{B}_E \sim \mu_o\sigma\,\frac{a}{4\pi}\sum_{m,n=o}^{\infty}\left(\frac{r}{a}\right)^m \int_\Omega P_m(\cos\gamma)\,\underset{\sim}{i}_{r'}\times \underset{\sim}{\zeta}_n(\phi',\theta')d\Omega. \qquad (52)$$

Since in (52) $r < a$, it follows that $\underset{\sim}{B}_E$ is at least O(a), as was to be demonstrated. Consequently, except for quantities that contribute terms of order a to the magnetic field

$$\underset{\sim}{J} - \underset{\sim}{J}_P \sim \sigma\,(\underset{\sim}{B}_o\cdot\nabla)(\underset{\sim}{\psi}-\underset{\sim}{\psi}_P). \qquad (53)$$

We must now examine the representation of the stream functions $\underset{\sim}{\psi}$ and $\underset{\sim}{\psi}_P$. Since $G_D \equiv G_{DP} + G_D - G_{DP}$, Eq. (24) may be written as follows:

$$\underset{\sim}{\psi} = \int_{V_{PS}} G_{DP}\,\underset{\sim}{\omega}\,dv + \int_{V_{PS}} \Delta G\underset{\sim}{\omega}dv, \qquad (54)$$

where ΔG is the difference between the Dirichlet Green's function for the region with boundaries P and S and the Dirichlet Green's function G_{DP} for the region with the boundary P alone. Similarly, we write the integral (12) as a sum of two parts:

$$\underset{\sim}{\psi}_P = \int_{V_P} G_{DP}\underset{\sim}{\omega}dv = \int_{V_{PS}} G_{DP}\underset{\sim}{\omega}dv + \int_{V_S} G_{DP}\underset{\sim}{\omega}dv. \tag{55}$$

Subtracting this from (54) we have

$$\underset{\sim}{\psi} - \underset{\sim}{\psi}_P = \int_{V_{PS}} \Delta G \underset{\sim}{\omega} dv - \int_{V_S} G_{DP}\underset{\sim}{\omega}dv. \tag{56}$$

It is convenient to express the various Green's functions in terms of a radius vector

$$\underset{\sim}{r}(x,y,z) = x\underset{\sim}{i}_x + y\underset{\sim}{i}_y + z\underset{\sim}{i}_z$$

and its image $\hat{\underset{\sim}{r}}$ in P,

$$\hat{\underset{\sim}{r}} = \underset{\sim}{r}\,(x,2D-y,z).$$

Clearly, on P

$$\underset{\sim}{r}|_P = \underset{\sim}{r}(x,D,z) = \hat{\underset{\sim}{r}}(x,D,z) = \hat{\underset{\sim}{r}}|_P \tag{57}$$

Thus,

$$G_{DP} = G_o(\underset{\sim}{r},\underset{\sim}{r}')-G_o(\hat{\underset{\sim}{r}},\underset{\sim}{r}'), \tag{58}$$

since (57) implies that G_{DP} defined by (58) vanishes on P, both terms on the right of (58) satisfy Laplace's equation, and G_{DP} has the same singularity as G_o when $\underset{\sim}{r}$ approaches $\underset{\sim}{r}'$ in V_P.

Making use of the expansion (29), one can obtain an estimate of the second term on the right side of (56) by observing that

$$\int_{V_S} G_{DP}\underset{\sim}{\omega} dv' = \frac{1}{4\pi}\sum_{n=0}^{\infty}\left[\frac{1}{r^{n+1}}\int_{V_S}\underset{\sim}{\omega} r'^n P_n(\cos\gamma)dv' - \frac{1}{\hat{r}^{n+1}}\int_{V_S}\underset{\sim}{\omega} r'^n P_n(\cos\hat{\gamma})dv'\right]$$

$$= \frac{a^2}{4\pi}\sum_{n=0}^{\infty}\frac{1}{n+3}\left[\left(\frac{a}{r}\right)^{n+1}\int_{\Omega}\underset{\sim}{\omega}_o P_n(\cos\gamma)d\Omega\right.,$$

$$\left. -\left(\frac{a}{\hat{r}}\right)^{n+1}\int_{\Omega}\underset{\sim}{\omega}_o P_n(\cos\hat{\gamma})d\Omega\right] + 0(a^3) \qquad (59)$$

$$= \frac{a^2}{3}\left(\frac{a}{r}\right)\underset{\sim}{\omega}_o + 0\,(a^3),$$

where $\underset{\sim}{\omega}_o$ is the value of $\underset{\sim}{\omega}$ at the center of the sphere and where γ is the angle between $\underset{\sim}{r}$ and $\underset{\sim}{r}'$ and $\hat{\gamma}$ is the angle between $\hat{\underset{\sim}{r}}$ and $\underset{\sim}{r}'$. Because of (59), the second term on the right side of (56) will be carried, simply, as a term of order a^2 in comparison with the first term whose order will also be obtained.

It is shown in Appendix A that, to the lowest order in $\frac{a}{D}$,

$$\Delta G \sim \frac{1}{4\pi}\left[-\frac{a}{\sqrt{a^4-2a^2rr'\cos\gamma+r^2r'^2}}+\frac{a}{\sqrt{a^4-2a^2\hat{r}r'\cos\hat{\gamma}+\hat{r}^2r'^2}}\right]. \qquad (60)$$

Since

$$\hat{r} > a$$

and in V_{PS}

$$r' \geq a,$$

ΔG can be expanded in the power series

$$\Delta G \sim \frac{a}{4\pi}\left[\frac{1}{\hat{r}r'}\sum_{n=0}^{\infty}\left(\frac{a^2}{\hat{r}r'}\right)^n P_n(\cos\hat{\gamma}) - \frac{1}{rr'}\sum_{n=0}^{\infty}\left(\frac{a^2}{rr'}\right)^n P_n(\cos\gamma)\right]. \qquad (61)$$

Hence,

$$\int_{V_{PS}} \Delta G \underset{\sim}{\omega} dv' \sim \frac{1}{4\pi} \sum_{n=0}^{\infty} a^n \left[\left(\frac{a}{\hat{r}}\right)^{n+1} \int_{V_{PS}} \frac{\underset{\sim}{\omega}}{r'^{\,n+1}} P_n(\cos \hat{\gamma}) dv' - \left(\frac{a}{r}\right)^{n+1} \int_{V_{PS}} \frac{\underset{\sim}{\omega}}{r'^{\,n+1}} P_n(\cos \gamma) dv' \right].$$

$$= \frac{1}{4\pi}\, a \left(\frac{1}{\hat{r}} - \frac{1}{r}\right) \int_{V_{PS}} \frac{\underset{\sim}{\omega}}{r'}\, dv' + O(a^2)$$

Hence, to the lowest order in a,

$$\underset{\sim}{\psi} - \underset{\sim}{\psi}_P \sim a \left(\frac{1}{\hat{r}} - \frac{1}{r}\right) \underset{\sim}{\mu}, \tag{62}$$

where $\underset{\sim}{\mu}$ is the constant vector given by

$$\underset{\sim}{\mu} = \frac{1}{4\pi} \int_{V_{PS}} \frac{\underset{\sim}{\omega}}{r'}\, dv'. \tag{63}$$

According to (53) and (62)

$$\underset{\sim}{J} - \underset{\sim}{J}_P \sim \sigma\, a \left(\frac{\underset{\sim}{B}_o \cdot \underset{\sim}{i}_r}{r^2} - \frac{\underset{\sim}{B}_o \cdot \underset{\sim}{i}_{\hat{r}}}{\hat{r}^2} \right) \underset{\sim}{\mu}, \tag{64}$$

where, as usual, the vectors $\underset{\sim}{i}_r$ and $\underset{\sim}{i}_{\hat{r}}$ are unit vectors in the directions indicated by their subscripts. Substituting (64) in (31) yields

$$\underset{\sim}{\alpha}_n \sim \frac{\sigma \mu_o a}{4\pi} \int_{V_{PS}} \left(\frac{\underset{\sim}{B}_o \cdot \underset{\sim}{i}_{r'}}{r'^{\,n+3}} - \frac{\underset{\sim}{B}_o \cdot \underset{\sim}{i}_{\hat{r}'}}{\hat{r}'^2\, r'^{\,n+1}} \right) P_n(\cos \gamma) dv'\, \underset{\sim}{\mu}. \tag{65}$$

Since terms of order $\frac{a}{D}$ are to be neglected, the integration region V_{PS} can be replaced by the region between S and a sphere with the same center and radius D. Then the term involving $\hat{r}'$ can be neglected as well. The result will be

$$\underset{\sim}{\alpha}_n \sim \frac{\sigma \mu_o a}{4\pi} \int_{\Omega} \int_a^D \frac{\underset{\sim}{B}_o \cdot \underset{\sim}{i}_{r'}}{r'^{\,n+1}} P_n(\cos \gamma) dr' d\Omega \underset{\sim}{\mu},$$

which, after neglecting terms of order $\frac{a}{D}$ becomes for* $n \neq 0$

$$a_n \sim \frac{\sigma\mu_o}{n4\pi a^{n-1}} \int_\Omega \underset{\sim}{B}_o \cdot \underset{\sim}{i}_{r'} P_n(\cos\gamma)d\Omega\, \underset{\sim}{\mu}. \tag{66}$$

The integral on the right side of (66) can be simplified by rotating coordinates, if necessary, so that the polar axis of the coordinate system corresponding to the integration variables has the direction $\underset{\sim}{i}_r$. Then the angle γ becomes θ'. If we express the geomagnetic field in the form $\underset{\sim}{B}_o = \underset{\sim}{i}_r B_{or} + \sqrt{B_o^2 - B_{or}^2}[\cos\beta\, \underset{\sim}{i}_\theta - \sin\beta\, \underset{\sim}{i}_\phi]$, the quantity $\underset{\sim}{B}_o \cdot \underset{\sim}{i}_{r'}$ may be written as follows:

$$\underset{\sim}{B}_o \cdot \underset{\sim}{i}_{r'} = \sqrt{B_o{}^2 - B_{or}{}^2}\, \sin\theta' \cos(\phi' + \beta) + B_{or}\cos\theta'.$$

Thus, (66) becomes

$$\underset{\sim}{\alpha}_n \sim \frac{\sigma\mu_o}{n4\pi a^{n-1}}\, \underset{\sim}{\mu} \int_0^{2\pi}\int_0^{\pi} \left(\sqrt{B_o{}^2 - B_{or}{}^2}\, \sin\theta' \cos(\phi' + \beta) + B_{or}\cos\theta'\right) P_n(\cos\theta') \sin\theta' d\theta' d\phi'$$

$$= \frac{\sigma\mu_o B_{or}}{n2a^{n-1}}\, \underset{\sim}{\mu} \int_0^{\pi} P_n(\cos\theta') \sin\theta' \cos\theta'\, d\theta'$$

$$= \frac{\sigma\mu_o B_{or}}{n2a^{n-1}}\, \underset{\sim}{\mu} \int_{-1}^{1} x\, P_n(x)dx = \begin{cases} \dfrac{\sigma\mu_o B_{or}}{3}\, \underset{\sim}{\mu}; & n = 1, \\ 0 & ; n \neq 1, \end{cases}$$

i.e.,

$$\underset{\sim}{\alpha}_1 \sim \frac{\sigma\mu_o}{3}(\underset{\sim}{i}_r \cdot \underset{\sim}{B}_o)\underset{\sim}{\mu}, \tag{67}$$

$$\underset{\sim}{\alpha}_n \sim 0,\ n \neq 1.$$

*$\underset{\sim}{\alpha}_o$ does not enter into the calculation of the magnetic field. See (32)

Then, according to (67),

$$\nabla \times \underset{\sim}{\alpha}_1 \sim \frac{\sigma\mu_o}{3} \nabla(\underset{\sim}{i}_r \cdot \underset{\sim}{B}_o) \times \underset{\sim}{\mu}, \tag{68}$$

$$\nabla \times \underset{\sim}{\alpha}_n \sim 0 \qquad , n \neq 1.$$

But

$$\begin{aligned} \nabla(\underset{\sim}{B}_o \cdot \underset{\sim}{i}_r) &= \frac{1}{r} \nabla(\underset{\sim}{B}_o \cdot \underset{\sim}{r}) - \frac{1}{r^3} (\underset{\sim}{B}_o \cdot \underset{\sim}{r})\underset{\sim}{r} \\ &= \frac{1}{r} (\underset{\sim}{B}_o - B_{or}\underset{\sim}{i}_r). \end{aligned} \tag{69}$$

From (68) and (69) it follows that

$$\underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{\alpha}_1 \sim \frac{\sigma\mu_o}{3r} \underset{\sim}{i}_r \times B_o \cdot \underset{\sim}{\mu} , \tag{70}$$

$$\underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{\alpha}_n \sim 0, \; n \neq 1.$$

Thus, for the boundary condition determined by (38), it follows from (70) and (41) that, to the lowest order in a,

$$\begin{aligned} b_a(\phi,\theta) \sim \frac{\partial \Phi}{\partial r}\Big|_{r=a} &\sim \underset{\sim}{i}_r \cdot \underset{\sim}{B}_P\Big|_{r=a} + \frac{\sigma\mu_o}{3} \underset{\sim}{i}_r \times \underset{\sim}{B}_o \cdot \underset{\sim}{\mu}. \\ &\sim \underset{\sim}{i}_r \cdot \underset{\sim}{B}_P(0) + \frac{\sigma\mu_o}{3} \underset{\sim}{i}_r \times \underset{\sim}{B}_o \cdot \underset{\sim}{\mu} \\ &= \underset{\sim}{i}_r \cdot [\underset{\sim}{B}_P(0) + \frac{\sigma\mu_o}{3} \underset{\sim}{B}_o \times \underset{\sim}{\mu}], \end{aligned} \tag{71}$$

where $\underset{\sim}{B}_P(0)$ is the unperturbed magnetic field at the center of the sphere. According to (71) the normal component of $\underset{\sim}{B}$ on S, to the lowest order in a, is equal to the normal component $\underset{\sim}{i}_r \cdot \underset{\sim}{\beta}$ of a constant vector $\underset{\sim}{\beta}$ defined by

$$\underset{\sim}{\beta} = \underset{\sim}{B}_P(0) + \frac{\sigma\mu_o}{3} \underset{\sim}{B}_o \times \underset{\sim}{\mu} .$$

The vector $\underset{\sim}{\beta}$ whose divergence and curl vanishes in S and whose normal component on S is $\underset{\sim}{i}_r \cdot \underset{\sim}{\beta}$ is the constant $\underset{\sim}{\beta}$, itself. This can be verified algebraically by expanding $b_a(\phi,\theta)$, given by (71), in a spherical harmonic series and substituting into (45). In this case the series consists of a single term $Z_1\ (\phi,\theta)$, the result of calculating the radial component of a constant vector. Thus, to the lowest order in a, we have the final result

$$\underset{\sim}{B} = \underset{\sim}{B}_P(0) + \frac{\sigma\mu_o}{3} \underset{\sim}{B}_o \times \underset{\sim}{\mu}, \tag{72}$$

where

$$\underset{\sim}{\mu} = \frac{1}{4\pi} \int_{V_{PS}} \frac{\underset{\sim}{\omega}}{\underset{\sim}{r}'} dv' \sim \frac{1}{4\pi} \int_{V_P} \frac{\underset{\sim}{\omega}}{r'} dv'.$$

Equation (72) states that the magnetic field inside a spherical buoy of small radius is spatially invariant and is given by the sum of the unperturbed field (as computed at the origin of the coordinate system of the sphere) and a correction term which is proportional to a volume integral comprising the vorticity function. The fact that the field inside the spherical cavity is spatially invariant is analogous to the situation that arises when a dielectric sphere is placed in an electrostatic field: there the field inside the sphere is also a constant. Indeed, the presence of the factor 1/3 in the correction term is reminiscent of the depolarization factor for the electrostatic field measured in a hollow spherical cavity in an extended dielectric medium. Thus, when n is the (relative) medium dielectric constant, the electrostatic field within a spherical cavity is

$$\underset{\sim}{E}_o + \frac{1}{3}\left(\frac{\kappa-1}{\kappa}\right) \underset{\sim}{E}_o$$

where $\underset{\sim}{E}_o$ is the incident (unperturbed) electrostatic field.

We now relate Equation (72) to the parameters of the internal wave field. We denote the cartesian coordinates with origin at the ocean surface by ξ, η, ζ where $\eta \leq 0$ is occupied by sea water. These coordinate axes are chosen parallel to the x,y,z coordinate employed in the preceding discussion (see Fig. 2). We denote the instantaneous vertical displacement due to internal wave motion by $q(\xi,\eta,\zeta,t)$. The horizontal coordinate pair (ξ,ζ) will also be denoted by the vector $\underset{\sim}{\kappa}$,

$$\underset{\sim}{\kappa} = \xi \, \underset{\sim}{i}_{\xi} + \zeta \, \underset{\sim}{i}_{\zeta} .$$

We now introduce the two-dimensional Fourier transform representation of q:

$$q(\underset{\sim}{\kappa},\eta,t) = \int_{-\infty}^{\infty}\!\!\int e^{-i\underset{\sim}{K}\cdot\underset{\sim}{\kappa}} \; \hat{q}(\underset{\sim}{K},\eta,t) d^2\underset{\sim}{K}, \tag{73}$$

where

$$\hat{q}(\underset{\sim}{K},\eta,t) = \sum_{n} \phi_n(\eta) \left[\frac{A_n^+(\underline{K})}{i\,\Omega_n} e^{i\Omega_n t} - \frac{A_n^-(\underline{K})}{i\,\Omega_n} e^{-i\Omega_n t} \right]. \tag{74}$$

The $\phi_n(\eta)$ are internal wave mode eigenfunctions satisfying

$$\frac{d^2}{d\eta^2} \phi_n(\eta) + K^2 \left(\frac{N^2(\eta)}{\Omega_n^{\;2}} - 1 \right) \phi_n(\eta) = 0, \tag{75}$$

with the normalization

$$\int_{-\infty}^{0} \phi_n(\eta) \; \phi_m(\eta) N^2(\eta) d\eta = \delta_{nm} . \tag{76}$$

A typical number of the sum in (74) can be interpreted as a simple harmonic internal wave with angular frequency Ω_n traveling in direction $\underset{\sim}{K}/K$, as indicated by the angle w in Fig. 2. Each component internal wave induces a corresponding partial magnetic field waveform which we denote by $\hat{\underset{\sim}{B}}^{(n)}(\underset{\sim}{K},\eta,t)$. The

actual field is then given by the superposition

$$\underset{\sim}{B}(\underset{\sim}{n},\eta,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\underset{\sim}{K}\cdot\underset{\sim}{\kappa}} \sum_{n} \hat{\underset{\sim}{B}}^{(n)}(\underset{\sim}{K},\eta,t)d^2\underset{\sim}{K} \ . \tag{77}$$

In the following, we shall be dealing explicitly only with a typical member $\hat{\underset{\sim}{B}}^{(n)}$, and use the notation $\hat{\underset{\sim}{B}}_P^{(n)}$ for the unperturbed field in the ocean. From [1], p. 54, one has

$$\hat{\underset{\sim}{B}}_P^{(n)} = \hat{\underset{\sim}{B}}_{PH}^{(n)} + \underset{\sim}{i}_\eta \ \hat{B}_{P\eta}^{(n)} \tag{78a}$$

$$\hat{\underset{\sim}{B}}_{PH}^{(n)} = \sigma\mu_o \left(\frac{\underset{\sim}{K}}{K}\right)\left[i\left(\frac{\phi_n}{K} - \frac{L_n^+}{2}\right)\underset{\sim}{i}_\eta + \left(\frac{L_n^-}{2}\right)\frac{\underset{\sim}{K}}{K}\right] \cdot \underset{\sim}{B}_o \ U_n(\underset{\sim}{K},t), \tag{78b}$$

$$\hat{B}_{P\eta}^{(n)} = -\sigma\mu_o\left[\left(\frac{L_n^-}{2}\right)\underset{\sim}{i}_\eta + i\left(\frac{L_n^+}{2}\right)\frac{\underset{\sim}{K}}{K}\right] \cdot \underset{\sim}{B}_o \ U_n(\underset{\sim}{K},t), \tag{78c}$$

where

$$U_n(\underset{\sim}{K},t) = A_n^+ \ (\underset{\sim}{K}) \ e^{i\Omega_n t} + A_n^- \ (\underset{\sim}{K}) \ e^{-i\Omega_n t}, \tag{78d}$$

$$L_n^+ = e^{-K\eta}\int_{-\infty}^{\eta} e^{K\eta} \ \phi_n(\eta')d\eta' + e^{K\eta}\int_{\eta}^{0} e^{-K\eta'}\phi_n(\eta')d\eta', \tag{78e}$$

$$L_n^- = e^{-K\eta}\int_{-\infty}^{\eta} e^{K\eta'}\phi_n(\eta')d\eta' - e^{K\eta}\int_{\eta}^{0} e^{-K\eta'} \ \phi_n(\eta')d\eta' . \tag{78f}$$

The subscript H in (78a) and (78b) is used to denote the fact that this vector is purely horizontal. These expressions give the unperturbed magnetic field below the ocean surface. We now obtain the perturbation due to the spherical enclosure, as given by the second term on the right of (72). We first compute $\underset{\sim}{\mu}$. When the location of the center of the sphere is expressed in terms of ξ,η,ζ ($\eta = -D < 0$) the integral may be written as follows:

$$\underset{\sim}{\mu} = \frac{1}{4\pi}\int_{-\infty}^{-\eta} dy \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\underset{\sim}{\omega}(x+\xi,\ y+\eta,\ z+\zeta)}{\sqrt{x^2 + y^2 + z^2}}\, dxdz$$

$$= \frac{1}{4\pi}\int_{\infty}^{0} dy' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\underset{\sim}{\omega}(\xi',\eta',\zeta')\ d\xi' d\zeta'}{\sqrt{(\eta'-\eta)^2+(\xi'-\xi)^2+(\zeta'-\zeta)^2}} .$$

Using the identity

$$\frac{1}{4\pi\sqrt{(\eta-\eta')^2 + (\xi-\xi')^2 + (\zeta-\zeta')^2}} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\underset{\sim}{K}\ \frac{e}{2K}^{-K|\eta-\eta'|-i\underset{\sim}{K}\cdot(\underline{\kappa}-\underline{\kappa}')}$$

we obtain

$$\underset{\sim}{\mu} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\underset{\sim}{K}\ \frac{e}{2K}^{-i\underset{\sim}{K}\cdot\underset{\sim}{\kappa}} \int_{-\infty}^{0} d\eta' \hat{\underset{\sim}{\omega}}(\eta',\underset{\sim}{K})\ e^{-K|\eta-\eta'|}, \tag{79}$$

where

$$\hat{\underset{\sim}{\omega}}(\eta',\underset{\sim}{K}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\underset{\sim}{\kappa}'\cdot\underset{\sim}{K}}\ \underset{\sim}{\omega}(\eta',\underset{\sim}{\kappa}')d^2\underset{\sim}{\kappa}'. \tag{80}$$

From [1] one finds

$$\hat{\underset{\sim}{\omega}}(\eta',\underset{\sim}{K}) = i(\underset{\sim}{i}_\eta \times \underset{\sim}{K})\sum_n \frac{N^2(\eta')}{\Omega_n^2}\ \phi_n(\eta')U_n(\underset{\sim}{K},t)$$

$$= \sum_n\ \hat{\underset{\sim}{\omega}}^{(n)}(\eta',\underset{\sim}{K}). \tag{81}$$

With the aid of (81), (79) may be written as follows:

$$\underset{\sim}{\mu} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\underset{\sim}{K}\ e^{-i\underset{\sim}{K}\cdot\underline{\kappa}} \sum_n\ \hat{\underset{\sim}{\mu}}_n ,$$

where

$$\underset{\sim}{\hat{\mu}}_n = i\,\frac{\underset{\sim}{i}_\eta \times \underset{\sim}{K}}{2K}\left(\frac{U_n(\underset{\sim}{K},t)}{\Omega_n^{\,2}}\right)\int_{-\infty}^{0} e^{-K|\eta-\eta'|}\, N^2(\eta')\phi_n(\eta')d\eta'.$$

From the differential equation (75) it may be shown that

$$\frac{1}{2\Omega_n^{\,2}}\int_{-\infty}^{0} e^{-K|\eta-\eta'|}\, N^2(\eta')\phi_n(\eta')d\eta' = \frac{\phi_n(\eta)}{K} - \frac{\phi_n(0)}{2K^2}\, e^{K\eta}. \qquad (82)$$

Thus, the correction term that must be added to (78) to account for the effect of the enclosure is given by

$$\frac{\sigma\mu_o}{3}\ \underset{\sim}{B}_o \times \underset{\sim}{\hat{\mu}}_n =$$

$$\frac{i\sigma\mu_o}{3}\ U_n(\underset{\sim}{K},t)\left[\left(\frac{\underset{\sim}{K}}{K}\cdot \underset{\sim}{B}_o\right)\underset{\sim}{i}_\eta - (\underset{\sim}{i}_\eta\cdot \underset{\sim}{B}_o)\,\frac{\underset{\sim}{K}}{K}\right]\left(\frac{\phi_n(\eta)}{K} - \frac{\dot{\phi}_n(0)e^{K\eta}}{2K^2}\right). \quad (83)$$

Comparing this expression with (78) we observe that, generally, the "correction" term is of the same order of magnitude as the unperturbed field.

V. THE MAGNETIC FIELD TO THE FIRST ORDER IN a

Since in the zeroth order approximation the magnetic field in the spherical buoy is constant, the (spatial) gradient of the magnetic field is zero in this approximation. To obtain a numerical estimate of the gradient, the magnetic field must be computed to the first order in a.

The magnetic field is determined by the currents $\underline{J}_M$ and $\underline{J}_E$ as indicated in (47). It is shown in Appendix B that the combination of $\underline{J}_E$ is $O(a^2)$. Consequently, to find the field to within $O(a)$ we need consider only $\underline{J}_M$. The magnetic field is determined by the $\underset{\sim}{\alpha}_n$ defined by (31), since it has already been shown that the integral over V_S contributes a term of order a^2. According to (53), (56), and (61) the current that determines the $\underset{\sim}{\alpha}_n$ depends upon vectors $\underset{\sim}{H}_n$ defined by

$$\underset{\sim}{H}_n = \int_{V_{PS}} \frac{\underset{\sim}{\omega}}{r'^{\,n+1}} \; P_n(\cos\gamma)dv'. \tag{84}$$

The vectors $\underset{\sim}{H}_n$ can be written in the form

$$\underset{\sim}{H}_n = \int_{\Omega} P_n(\cos\gamma) \; \underset{\sim}{F}_n(\phi',\theta')d\Omega, \tag{85}$$

where

$$\underset{\sim}{F}_n(\phi,\theta) = \int_a^{D\sec\theta} \frac{\underset{\sim}{\omega}}{r'^{\,n-1}}\, dr'. \tag{86}$$

It is clear from (86) that for n = 0 or 1 $\underset{\sim}{F}_n$ is O(1) in a, but for n = 2 $\underset{\sim}{F}_n$ is O(log a). For larger values of n, i.e., n > 2, $\underset{\sim}{F}_n$ is O (a^{2-n}).

Each cartesian component of $\underset{\sim}{F}_n$ can be expanded in a surface spherical harmonic series. The results leads to a vector expansion

$$\underset{\sim}{F}_n(\phi,\theta) = \sum_{m=0}^{\infty} \underset{\sim}{Z}_{mn}(\phi,\theta), \tag{87}$$

where $\underset{\sim}{Z}_{mn}$ is a vector each of whose cartesian components is a spherical harmonic of degree m. It follows from (85) and well known properties of surface spherical harmonics (cf. MacRobert, Ch. VII) that

$$\underset{\sim}{H}_n = \frac{4\pi}{2n+1} \underset{\sim}{Z}_{nn}(\phi,\theta) = \frac{4\pi}{2n+1} \underset{\sim}{Z}_n(\phi,\theta), \tag{88}$$

where $\underset{\sim}{Z}_n$ has been written in place of $\underset{\sim}{Z}_{nn}$.

From (56) and (62), neglecting terms of order a^2 and of order $\frac{a}{D}$, it follows that

$$\underset{\sim}{\psi} - \underset{\sim}{\psi}_P \sim -\sum_{n=0}^{\infty} \frac{a^n}{2n+1} \left(\frac{a}{r}\right)^{n+1} \underset{\sim}{Z}_n(\phi,\theta). \tag{89}$$

From (53) it follows that

$$\underset{\sim}{J} - \underset{\sim}{J}_P \sim \sigma \sum_{n=0}^{\infty} \frac{a^{2n+1}}{r^{n+2}} \underset{\sim}{Y}_{n+1}(\phi,\theta), \tag{90}$$

where

$$\underset{\sim}{Y}_{n+1}(\phi,\theta) = \frac{1}{2n+1} [(n+1)(\underset{\sim}{i}_r \cdot \underset{\sim}{B}_o)\underset{\sim}{Z}_n(\phi,\theta) - (\underset{\sim}{B}_o \cdot \nabla_\Omega)\underset{\sim}{Z}_n(\phi,\theta)], \tag{91}$$

is a vector each of whose cartesian components is a surface spherical harmonic of degree n+1. This last statement follows from the fact that each component of each term in the series (89) for $\underset{\sim}{\psi} - \underset{\sim}{\psi}_P$ separately satisfies Laplace's equation and still does so after the operator $(\underset{\sim}{B}_o \cdot \nabla)$ is applied.

According to (31), then,

$$\underset{\sim}{\alpha}_n \sim \frac{\sigma\mu_o}{4\pi} \sum_{m=o}^{\infty} a^{2m+1} \int_{V_{PS}} \frac{\underset{\sim}{Y}_{m+1}(\phi',\theta')}{r'^{m+n+3}} P_n(\cos\gamma)dv'$$

$$= \frac{\sigma\mu_o}{4\pi} \sum_{m=o}^{\infty} a^{2m+1} \int_{\Omega} \underset{\sim}{Y}_{m+1}(\phi',\theta')P_n(\cos\gamma) \int_a^{D\sec\theta} \frac{dr'}{r'^{m+n+1}} d\Omega$$

$$\sim \frac{\sigma\mu_o}{4\pi} \sum_{m=o}^{\infty} \frac{a^{m-n+1}}{m+n} \int_{\Omega} \underset{\sim}{Y}_{m+1}(\phi',\theta')P_n(\cos\gamma)d\Omega$$

$$= \frac{\sigma\mu_o}{4n^2-1} \underset{\sim}{Y}_n(\phi,\theta), \qquad (92)$$

wherein terms of order $\frac{a}{D}$ have been dropped and the same properties of spherical harmonics used in deriving (88) have been applied. It should be noted that for $n = 1$ or 2 $\underset{\sim}{Y}_n$ is $0(1)$ in a, for $n = 3$ $\underset{\sim}{Y}_n$ is 0 (log a), and for $n > 3$ $\underset{\sim}{Y}_n$ is $0(a^{3-n})$.

According to (88)

$$\underset{\sim}{Z}_1(\phi,\theta) = \frac{3}{4\pi} \underset{\sim}{H}_1, \qquad (93)$$

and therefore, according to (84)

$$\underset{\sim}{Z}_1(\phi,\theta) = \frac{3}{4\pi} \int_{V_{PS}} \frac{\underset{\sim}{\omega}}{r'^2} \cos\gamma\, dv' \sim \frac{3}{4\pi} \int_{V_P} \frac{\underset{\sim}{\omega}}{r'^2} \cos\gamma\, dv'. \qquad (94)$$

Thus,

$$\underset{\sim}{Z}_1(\phi,\theta) \sim \underset{\sim}{\mu}_3 \cos\theta + (\underset{\sim}{\mu}_1 \cos\phi + \underset{\sim}{\mu}_2 \sin\phi) \sin\theta, \qquad (95)$$

where

$$\underset{\sim}{\mu}_3 = \frac{3}{4\pi} \int_{V_P} \frac{\underset{\sim}{\omega}}{r'^2} \cos\theta'\, dv', \qquad (96)$$

$$\underset{\sim}{\mu}_1 = \frac{3}{4\pi}\int_{V_p} \frac{\underset{\sim}{\omega}}{r'^2} \cos \phi' \sin \theta' \, dv', \tag{97}$$

and

$$\underset{\sim}{\mu}_2 = \frac{3}{4\pi}\int_{V_p} \frac{\underset{\sim}{\omega}}{r'^2} \sin \phi' \sin \theta' \, dv'. \tag{98}$$

Then

$$\begin{aligned}\underset{\sim}{Y}_2 = \frac{1}{3} \Big\{ & \left[2(\underset{\sim}{i}_r \cdot \underset{\sim}{B}_o)\cos \theta + (\underset{\sim}{i}_\theta \cdot \underset{\sim}{B}_o)\sin \theta\right]\underset{\sim}{\mu}_3 \\ & + \Big[2(\underset{\sim}{i}_r \cdot \underset{\sim}{B}_o) \cos \phi \sin \theta - (\underset{\sim}{i}_\theta \cdot \underset{\sim}{B}_o)\cos \phi \cos \theta \\ & + (\underset{\sim}{i}_\phi \cdot \underset{\sim}{B}_o)\sin \phi\Big] \underset{\sim}{\mu}_1 + \Big[2(\underset{\sim}{i}_r \cdot \underset{\sim}{B}_o)\sin \phi \sin \theta \\ & - (\underset{\sim}{i}_\theta \cdot \underset{\sim}{B}_o) \sin \phi \cos \theta - (\underset{\sim}{i}_\phi \cdot \underset{\sim}{B}_o)\cos \phi\Big] \underset{\sim}{\mu}_2. \end{aligned} \tag{99}$$

According to (38), if the boundary term $b_a(\phi,\theta)$ is supplemented by a quantity $\Delta\beta$ given by

$$\Delta\beta = (\underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{\alpha}_2)_{r=a} \, a^2, \tag{100}$$

because of (92) and the order in a of $\underset{\sim}{Y}_n$ for $n > 2$, the terms neglected in $b_a(\phi,\theta)$ will be $0\,(a^3 \log a)$ and higher. In order to obtain the complete field correct to terms of order a, however, it is also necessary to replace the quantity $\underset{\sim}{B}_P(0)$ used in (71) and (72) by $\underset{\sim}{B}_P|_{r=a}$ correct to the first order in a. Thus, the new boundary value $b_a(\phi,\theta)$ is given by

$$\begin{aligned} b_a(\phi,\theta) \sim \underset{\sim}{i}_r \cdot \underset{\sim}{B}_P(0) + \frac{\sigma\mu_o}{3}\, \underset{\sim}{i}_r \cdot \underset{\sim}{B}_o \times \underset{\sim}{\mu} + a\, \underset{\sim}{i}_r \cdot \frac{\partial \underset{\sim}{B}_P}{\partial r}\Big|_{r=0} + \frac{\sigma\mu_o}{15}\, a^2 \underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{Y}_2(\phi,\theta)\Big|_{r=a} \\ + 0\,(a^3 \log a), \end{aligned} \tag{101}$$

wherein the fourth term on the right side of the equation results from (92) and (100).

The fourth term on the right side of (101) is actually $O(a)$ rather than $O(a^2)$ because of the curl operation. The expression

$$\underset{\sim}{r} \times \nabla \cdot \left[r^2 \underset{\sim}{Y}_2(\phi\ \theta)\right] = r^3 \underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{Y}_2(\phi,\theta) \qquad (102)$$

is a homogeneous polynomial of degree two in the cartesian variables x, y, and z because the cartesian components of $\underset{\sim}{Y}_2(\phi,\theta)$ are all surface spherical harmonics of degree two. It then follows (cf. Ref. 3, p. 140) that the fourth term on the right side of (101) can be expressed as the sum of a surface spherical harmonic of degree two and a surface spherical harmonic of degree zero, i.e., a constant. That constant is the integral over the unit sphere of the original term, for which according to Gauss' theorem

$$\int_S \underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{Y}_2 d\Omega = \int_{V_S} \nabla \cdot (\nabla \times \underset{\sim}{Y}_2) dv = 0. \qquad (103)$$

Hence the original term is a surface spherical harmonic of degree two.

The third term on the right side of (101) can be written

$$a\, \underset{\sim}{i}_r \cdot \frac{\partial \underset{\sim}{B}_P}{\partial r}\Big|_{r=0} = a\left(\underset{\sim}{i}_r \cdot \nabla\right)\underset{\sim}{B}_P\Big|_{r=0}$$

$$= a\left[\left(\underset{\sim}{i}_r \cdot \frac{\partial \underset{\sim}{B}_P}{\partial x}\Big|_{r=0}\right)\left(\underset{\sim}{i}_r \cdot \underset{\sim}{i}_x\right) + \left(\underset{\sim}{i}_r \cdot \frac{\partial \underset{\sim}{B}_P}{\partial y}\Big|_{r=0}\right)\left(\underset{\sim}{i}_r \cdot \underset{\sim}{i}_y\right)\right.$$

$$\left. + \left(\underset{\sim}{i}_r \cdot \frac{\partial \underset{\sim}{B}_P}{\partial z}\Big|_{r=0}\right)\left(\underset{\sim}{i}_r \cdot \underset{\sim}{i}_z\right)\right]$$

$$= a\left[B_2(\phi,\theta) + \frac{1}{3}\left(\nabla \cdot \underset{\sim}{B}_P\right)_{r=0}\right] = a\, B_2(\phi,\theta), \qquad (104)$$

where $B_2(\phi,\theta)$ can be recognized as a surface spherical harmonic of degree two because of the identity (B-4). Here, of course, $B_2(\phi,\theta)$ is given by

$$B_2(\phi,\theta) = \underset{\sim}{i}_r \cdot \left.\frac{\partial \underset{\sim}{B}_P}{\partial r}\right|_{r=0} . \tag{105}$$

These considerations make it possible to write an expression, correct to terms of order $a^3 \log a$, for Φ, the function that satisfies Laplace's equation subject to the condition that on S its radial derivative is equal to the function $b_a(\phi,\theta)$ given by (101). Since the magnetic field inside the sphere is $\nabla\Phi$, it is then given approximately there by

$$\underset{\sim}{B} \sim \underset{\sim}{B}_P(0) + \frac{\sigma\mu_o}{3} \underset{\sim}{B}_o \times \underset{\sim}{\mu} + \frac{\sigma\mu_o ar}{15} \left\{ \left(\underset{\sim}{i}_r + \frac{1}{2}\nabla_\Omega\right)\left[\underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{Y}_2(\phi,\theta)\right]_{r=a} \right\}$$
$$+ r\left(\underset{\sim}{i}_r + \frac{1}{2}\nabla_\Omega\right) B_2(\phi,\theta), \tag{106}$$

correct to terms of order a.

Because of (104) it follows that

$$B_2(\phi,\theta) = \frac{x(\underset{\sim}{\beta}_1 \cdot r) + z(\underset{\sim}{\beta}_2 \cdot \underset{\sim}{r}) + y(\underset{\sim}{\beta}_3 \cdot \underset{\sim}{r})}{r^2} , \tag{107}$$

where

$$\underset{\sim}{\beta}_1 = \left.\frac{\partial \underset{\sim}{B}_P}{\partial x}\right|_{r=0}, \quad \underset{\sim}{\beta}_2 = \left.\frac{\partial \underset{\sim}{B}_P}{\partial z}\right|_{r=0}, \quad \underset{\sim}{\beta}_3 = \left.\frac{\partial \underset{\sim}{B}_P}{\partial y}\right|_{r=0} . \tag{108}$$

Then, according to (106), the contribution of $B_2(\phi,\theta)$ to $\underset{\sim}{B}$ can be written

$$r\left(\underset{\sim}{i}_r + \frac{1}{2}\nabla_\Omega\right) B_2(\phi,\theta) = r\left(\underset{\sim}{i}_r + \frac{1}{2}\nabla_\Omega\right)\left[\frac{\left(x\underset{\sim}{\beta}_1 + z\,\underset{\sim}{\beta}_2 + y\underset{\sim}{\beta}_3\right)\cdot \underset{\sim}{r}}{r^2}\right]. \tag{109}$$

The vector operator in (109) can be written

$$\left(\underset{\sim}{i}_r + \frac{1}{2}\nabla_\Omega\right) = \frac{1}{2}\, r\, \nabla + \underset{\sim}{i}_r \left(\frac{3}{2r} - \frac{1}{2}\frac{\partial}{\partial r}\right) r$$

$$= \frac{1}{2} r \nabla + \frac{\underset{\sim}{r}}{r^2}\left(\frac{3}{2} - \frac{1}{2} \underset{\sim}{r} \cdot \nabla\right) r$$

$$= \left[\frac{1}{2} \nabla + \frac{\underset{\sim}{r}}{r^2} \left(1 - \frac{1}{2} \underset{\sim}{r} \cdot \nabla\right)\right] r. \tag{110}$$

It is now useful to observe that relations such as

$$\nabla \left[\frac{x(\underset{\sim}{\beta}_1 \cdot \underset{\sim}{r})}{r}\right] = \frac{\underset{\sim}{i}_x(\underset{\sim}{\beta}_1 \cdot \underset{\sim}{r})}{r} + \frac{x\underset{\sim}{\beta}_1}{r} - \frac{x(\underset{\sim}{\beta}_1 \cdot \underset{\sim}{r})}{r^3} \underset{\sim}{r} \tag{111}$$

and thus, also,

$$\underset{\sim}{r} \cdot \nabla \left[\frac{x(\underset{\sim}{\beta}_1 \cdot \underset{\sim}{r})}{r}\right] = \frac{x(\underset{\sim}{\beta}_1 \cdot \underset{\sim}{r})}{r} \tag{112}$$

hold.

Relations similar to (111) and (112) can be used to evaluate (110). Thus, it follows that

$$\begin{aligned} \hat{r}\left(\underset{\sim}{i}_r + \frac{1}{2} \nabla_\Omega\right) B_2(\phi,\theta) = \frac{1}{2}\Bigg\{&\left[\underset{\sim}{i}_x\underset{\sim}{r} + \left(\underset{\sim}{i}_x \cdot \underset{\sim}{r}\right) \underset{\approx}{I}\right]\cdot\underset{\sim}{\beta}_1 \\ &+ \left[\underset{\sim}{i}_z\underset{\sim}{r} + \left(\underset{\sim}{i}_z \cdot \underset{\sim}{r}\right)\underset{\approx}{I}\right] \cdot\underset{\sim}{\beta}_2 \\ &+ \left[\underset{\sim}{i}_y\underset{\sim}{r} + \left(\underset{\sim}{i}_y \cdot \underset{\sim}{r}\right)\underset{\approx}{I}\right] \cdot\underset{\sim}{\beta}_3\Bigg\}, \end{aligned} \tag{113}$$

where $\underset{\approx}{I}$ is the unit dyadic defined by

$$\underset{\approx}{I} = \underset{\sim}{i}_x\underset{\sim}{i}_x + \underset{\sim}{i}_y\underset{\sim}{i}_y + \underset{\sim}{i}_z\underset{\sim}{i}_z. \tag{114}$$

It follows from (91) and (95) that, except for an additive constant,

$$\underset{\sim}{Y}_2(\phi,\theta) = U_1(\phi,\theta)\underset{\sim}{\mu}_1 + U_2(\phi,\theta)\underset{\sim}{\mu}_2 + U_3(\phi,\theta)\underset{\sim}{\mu}_3, \tag{115}$$

where

$$U_1(\phi,\theta) = \frac{4}{5}\,\frac{x(\underset{\sim}{B}_o \cdot \underset{\sim}{r})}{r^2}\ ,\ U_2(\phi,\theta) = \frac{4}{5}\,\frac{z(\underset{\sim}{B}_o \cdot \underset{\sim}{r})}{r^2}\ , \tag{116}$$

$$U_3(\phi,\theta) = \frac{4}{5}\,\frac{y(\underset{\sim}{B}_o \cdot \underset{\sim}{r})}{r^2}\ . \tag{117}$$

From (115) it follows that

$$\underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{Y}_2 = \frac{1}{r}\left(\underset{\sim}{r} \times \nabla\, U_1 \cdot \underset{\sim}{\mu}_1 + \underset{\sim}{r} \times \nabla\, U_2 \cdot \underset{\sim}{\mu}_2 + \underset{\sim}{r} \times \nabla\, U_3 \cdot \underset{\sim}{\mu}_3\right). \tag{118}$$

From (116) it will be found that

$$\underset{\sim}{r} \times \nabla\, U_1 = \frac{4}{5}\left[\left(\frac{\underset{\sim}{B}_o \cdot \underset{\sim}{r}}{r^2}\,\underset{\sim}{r} \times \underset{\sim}{i}_x + \frac{x\, \underset{\sim}{r} \times \underset{\sim}{B}_o}{r^2}\right)\right]. \tag{119}$$

Similar relations hold, of course, for $\underset{\sim}{r} \times \nabla U_2$ *and* $\underset{\sim}{r} \times \nabla U_3$. It then follows that

$$\left[r\, \underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{Y}_2\right]_{r=a} = r\, \underset{\sim}{i}_r \cdot \nabla \times \underset{\sim}{Y}_2\ . \tag{120}$$

From (110), (119), and (120) it follows that

$$\left(\underset{\sim}{i}_r + \frac{1}{2}\nabla_\Omega\right)\left(\underset{\sim}{i}_r \cdot \nabla\times\underset{\sim}{Y}_2\right)_{r=a} = \frac{4}{5}\left[\frac{1}{2}\nabla + \frac{\underset{\sim}{r}}{r^2}\left(1 - \frac{1}{2}\,\underset{\sim}{r}\cdot\nabla\right)\right]\frac{r}{a}\left\{\left[\left(\frac{\underset{\sim}{B}_o \cdot \underset{\sim}{r}}{r^2}\right)\underset{\sim}{r} \times \underset{\sim}{i}_x\right.\right.$$

$$\left. + \frac{x\underset{\sim}{r}\times\underset{\sim}{B}_o}{r^2}\right]\cdot \underset{\sim}{\mu}_1 + \left[\left(\frac{\underset{\sim}{B}_o \cdot \underset{\sim}{r}}{r^2}\right)\underset{\sim}{r}\times\underset{\sim}{i}_z + \frac{z\underset{\sim}{r}\times\underset{\sim}{B}_o}{r^2}\right]\cdot \underset{\sim}{\mu}_2$$

$$\left. + \left[\left(\frac{\underset{\sim}{B}_o \cdot \underset{\sim}{r}}{r^2}\right)\underset{\sim}{r} \times \underset{\sim}{i}_y + \frac{y\, \underset{\sim}{r} \times \underset{\sim}{B}_o}{r^2}\right]\cdot \underset{\sim}{\mu}_3\right\}$$

$$= \frac{2}{5a}\left(\nabla + \frac{\underset{\sim}{r}}{r^2}\right)\left\{\left[\left(\frac{\underset{\sim}{B}_o\cdot\underset{\sim}{r}}{r}\right)\underset{\sim}{r} \times \underset{\sim}{i}_x + \frac{x\, \underset{\sim}{r} \times \underset{\sim}{B}_o}{r}\right]\cdot \underset{\sim}{\mu}_1 + \left[\left(\frac{\underset{\sim}{B}_o\cdot\underset{\sim}{r}}{r}\right)\underset{\sim}{r}\times\underset{\sim}{i}_z + \frac{z\underset{\sim}{r}\times\underset{\sim}{B}_o}{r}\right]\cdot\underset{\sim}{\mu}_2\right.$$

$$+\left[\left(\frac{\underset{\sim}{B}_o \cdot \underset{\sim}{r}}{r}\right)\underset{\sim}{r} \times \underset{\sim}{i}_y + \frac{y\, \underset{\sim}{r} \times \underset{\sim}{B}_o}{r}\right] \cdot \underset{\sim}{\mu}_3\Bigg\} \qquad (121)$$

$$= \frac{2}{5ar}\left\{\left[\underset{\sim}{r} \cdot \left(\underset{\sim}{i}_x \times \underset{\sim}{\mu}_1 + \underset{\sim}{i}_z \times \underset{\sim}{\mu}_2 + \underset{\sim}{i}_y \times \underset{\sim}{\mu}_3\right)\right]\underset{\sim}{B}_o + \left(\underset{\sim}{B}_o \cdot \underset{\sim}{r}\right)\left(\underset{\sim}{i}_x \times \underset{\sim}{\mu}_1 + \underset{\sim}{i}_z \times \underset{\sim}{\mu}_2 + \underset{\sim}{i}_y \times \underset{\sim}{\mu}_3\right)\right.$$

$$+\left[\left(\underset{\sim}{r} \cdot \underset{\sim}{B}_o \times \underset{\sim}{\mu}_1\right)\underset{\sim}{i}_x + x\left(\underset{\sim}{B}_o \times \underset{\sim}{\mu}_1\right) + \left(\underset{\sim}{r} \cdot \underset{\sim}{B}_o \times \underset{\sim}{\mu}_2\right)\underset{\sim}{i}_z + z\left(\underset{\sim}{B}_o \times \underset{\sim}{\mu}_2\right)\right.$$

$$\left.\left. + \left(\underset{\sim}{r} \cdot \underset{\sim}{B}_o \times \underset{\sim}{\mu}_3\right)\underset{\sim}{i}_y + y\left(\underset{\sim}{B}_o \times \underset{\sim}{\mu}_3\right)\right]\right\}$$

$$= \frac{2}{5ar}\left\{\left[\underset{\sim}{B}_o\left(\underset{\sim}{r} \times \underset{\sim}{i}_x\right) + \underset{\sim}{i}_x\left(\underset{\sim}{r} \times \underset{\sim}{B}_o\right) + \left(\underset{\sim}{i}_x \cdot \underset{\sim}{r}\right)\left(\underset{\approx}{I} \times \underset{\sim}{B}_o\right) + \left(\underset{\sim}{B}_o \cdot \underset{\sim}{r}\right)\underset{\approx}{J}_1\right] \cdot \underset{\sim}{\mu}_1\right.$$

$$+ \left[\underset{\sim}{B}_o\left(\underset{\sim}{r} \times \underset{\sim}{i}_z\right) + \underset{\sim}{i}_z\left(\underset{\sim}{r} \times \underset{\sim}{B}_o\right) + \left(\underset{\sim}{i}_z \cdot \underset{\sim}{r}\right)\left(\underset{\approx}{I} \times \underset{\sim}{B}_o\right) + \left(\underset{\sim}{B}_o \cdot \underset{\sim}{r}\right)\underset{\approx}{J}_2\right] \cdot \underset{\sim}{\mu}_2 \qquad (121)$$

$$\left. + \left[\underset{\sim}{B}_o\left(\underset{\sim}{r} \times \underset{\sim}{i}_y\right) + \underset{\sim}{i}_y\left(\underset{\sim}{r} \times \underset{\sim}{B}_o\right) + \left(\underset{\sim}{i}_y \cdot \underset{\sim}{r}\right)\left(\underset{\approx}{I} \times \underset{\sim}{B}_o\right) + \left(\underset{\sim}{B}_o \cdot \underset{\sim}{r}\right)\underset{\approx}{J}_3\right] \cdot \underset{\sim}{\mu}_3 \right. \qquad (121a)$$

where

$$\underset{\approx}{J}_1 = \underset{\sim}{i}_z\underset{\sim}{i}_y - \underset{\sim}{i}_y\underset{\sim}{i}_z, \quad \underset{\approx}{J}_2 = \underset{\sim}{i}_y\underset{\sim}{i}_x - \underset{\sim}{i}_x\underset{\sim}{i}_y, \quad \underset{\approx}{J}_3 = \underset{\sim}{i}_x\underset{\sim}{i}_z - \underset{\sim}{i}_z\underset{\sim}{i}_x.$$

Expressions (113) and (121) can be substituted into (106) to obtain an explicit expression for the magnetic field. Then, with

$$\underset{\sim}{\beta}_o = \underset{\sim}{B}_P(0),$$

the magnetic field can be written:

$$\underset{\sim}{B} = \left[\underset{\sim}{\beta}_o + \frac{\sigma\mu_o}{3}\left(\underset{\sim}{B}_o \times \underset{\sim}{\mu}\right) + \frac{1}{2}\left\{\left[\underset{\sim}{i}_x\underset{\sim}{r} + x\,\underset{\approx}{I}\right] \cdot \underset{\sim}{\beta}_1 + \left[\underset{\sim}{i}_z\underset{\sim}{r} + z\underset{\approx}{I}\right] \cdot \underset{\sim}{\beta}_2\right.\right.$$

$$
\begin{aligned}
&+\left[\mathbf{i}_y\mathbf{r} + y\mathbf{I}\right] \cdot \boldsymbol{\beta}_3\Big\} + \frac{2\sigma\mu_o}{75}\Big\{\Big[\mathbf{B}_o\left(\mathbf{r} \times \mathbf{i}_x\right) + \mathbf{i}_x\left(\mathbf{r} \times \mathbf{B}_o\right) + x\mathbf{I}\times\mathbf{B}_o \\
&\qquad + \left(\mathbf{B}_o \cdot \mathbf{r}\right)\mathbf{J}_1\Big] \cdot \boldsymbol{\mu}_1 + \Big[\mathbf{B}_o\left(\mathbf{r} \times \mathbf{i}_z\right) \\
&\qquad + \mathbf{i}_z\left(\mathbf{r} \times \mathbf{B}_o\right) + z\mathbf{I} \times \mathbf{B}_o + \left(\mathbf{B}_o \cdot \mathbf{r}\right)\mathbf{J}_2\Big] \cdot \boldsymbol{\mu}_2 \\
&\qquad + \Big[\mathbf{B}_o\left(\mathbf{r} \times \mathbf{i}_y\right) + \mathbf{i}_y\left(\mathbf{r} \times \mathbf{B}_o\right) + y\mathbf{I} \times \mathbf{B}_o \\
&\qquad + \left(\mathbf{B}_o \cdot \mathbf{r}\right)\mathbf{J}_3\Big] \cdot \boldsymbol{\mu}_3\Big\} . \qquad (122)
\end{aligned}
$$

The foregoing analysis shows that terms neglected in the approximations for the magnetic field and its gradient are of the same order as a^n multiplied by the quantities H_n given by (84). That is, the terms neglected are of the order of a^n times higher derivatives of the quantity

$$\boldsymbol{\mu}(\xi,\zeta,\eta) = \int_{V_{PS}} G_o \boldsymbol{\omega} dv' .$$

As indicated in (79), the convolution theorem implies that the horizontal Fourier transform $\hat{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ has the form

$$\hat{\boldsymbol{\mu}}(\mathbf{K}) = \int_{-\infty}^{0} \frac{\hat{\boldsymbol{\omega}}(\eta',\mathbf{K})}{2K}\, e^{-i\,\mathbf{K}\,\cdot\,\boldsymbol{\kappa}\,-\,K|\eta-\eta'|}\, d\eta$$

It follows that an n^{th} order derivative of $\boldsymbol{\mu}$ has a Fourier transform that is proportional to $K^n\hat{\boldsymbol{\mu}}(\mathbf{K})$. Thus, a condition that might be imposed, justifying the magnetic field approximation used in this paper, can be stated as follows: the vorticity spectrum is negligible except for the wave number region in which $Ka \ll 1$. Because of the monotonic relation between frequency and wave number, this condition is consistent with the static approximation used in calculating the magnetic field.

VI. THE MAGNETIC FIELD GRADIENT

The magnetic field gradient G_{pq} relative to unit vectors $\underset{\sim}{1}_p$ and $\underset{\sim}{1}_q$ is defined by

$$G_{pq} = G_{qp} = \underset{\sim}{1}_p \cdot \nabla\left(\underset{\sim}{1}_q \cdot \underset{\sim}{B}\right). \tag{123}$$

If (123) is applied to (122) the result is

$$G_{pq} = \frac{1}{2}\left(\underset{\sim}{\Lambda}_1 \cdot \underset{\sim}{\beta}_1 + \underset{\sim}{\Lambda}_2 \cdot \underset{\sim}{\beta}_2 + \underset{\sim}{\Lambda}_3 \cdot \underset{\sim}{\beta}_3\right) + \frac{2\sigma\mu_o}{75}\left(\underset{\sim}{\Gamma}_1 \cdot \underset{\sim}{\mu}_1 + \underset{\sim}{\Gamma}_2 \cdot \underset{\sim}{\mu}_2 + \underset{\sim}{\Gamma}_3 \cdot \underset{\sim}{\mu}_3\right), \tag{124}$$

where

$$\underset{\sim}{\Lambda}_1 = \left(\underset{\sim}{1}_q \cdot \underset{\sim}{i}_x\right)\underset{\sim}{1}_p + \left(\underset{\sim}{1}_p \cdot \underset{\sim}{i}_x\right)\underset{\sim}{1}_q,$$

$$\underset{\sim}{\Lambda}_2 = \left(\underset{\sim}{1}_q \cdot \underset{\sim}{i}_z\right)\underset{\sim}{1}_p + \left(\underset{\sim}{1}_p \cdot \underset{\sim}{i}_z\right)\underset{\sim}{1}_q,$$

$$\underset{\sim}{\Lambda}_3 = \left(\underset{\sim}{1}_q \cdot \underset{\sim}{i}_y\right)\underset{\sim}{1}_p + \left(\underset{\sim}{1}_p \cdot \underset{\sim}{i}_y\right)\underset{\sim}{1}_q,$$

$$\underset{\sim}{\Gamma}_1 = \left(\underset{\sim}{1}_q \cdot \underset{\sim}{B}_o\right)\underset{\sim}{1}_p \times \underset{\sim}{i}_x + \left(\underset{\sim}{1}_q \cdot \underset{\sim}{i}_x\right)\underset{\sim}{1}_p \times \underset{\sim}{B}_o + \left(\underset{\sim}{1}_p \cdot \underset{\sim}{i}_x\right)\underset{\sim}{1}_q \times \underset{\sim}{B}_o + \left(\underset{\sim}{1}_p \cdot \underset{\sim}{B}_o\right)\underset{\sim}{1}_q \times \underset{\sim}{i}_x,$$

$$\underset{\sim}{\Gamma}_2 = \left(\underset{\sim}{1}_q \cdot \underset{\sim}{B}_o\right)\underset{\sim}{1}_p \times \underset{\sim}{i}_z + \left(\underset{\sim}{1}_q \cdot \underset{\sim}{i}_z\right)\underset{\sim}{1}_p \times \underset{\sim}{B}_o + \left(\underset{\sim}{1}_p \cdot \underset{\sim}{i}_z\right)\underset{\sim}{1}_q \times \underset{\sim}{B}_o + \left(\underset{\sim}{1}_p \cdot \underset{\sim}{B}_o\right)\underset{\sim}{1}_q \times \underset{\sim}{i}_z,$$

$$\underset{\sim}{\Gamma}_3 = \left(\underset{\sim}{1}_q \cdot \underset{\sim}{B}_o\right)\underset{\sim}{1}_p \times \underset{\sim}{i}_y + \left(\underset{\sim}{1}_q \cdot \underset{\sim}{i}_y\right)\underset{\sim}{1}_p \times \underset{\sim}{B}_o + \left(\underset{\sim}{1}_p \cdot \underset{\sim}{i}_y\right)\underset{\sim}{1}_q \times \underset{\sim}{B}_o + \left(\underset{\sim}{1}_p \cdot \underset{\sim}{B}_o\right)\underset{\sim}{1}_q \times \underset{\sim}{i}_y.$$

Part of the expression (124) for the magnetic field gradient depends upon the vectors $\underset{\sim}{\beta}_i$ which are related to the gradient G^P_{pq} of the magnetic field in the absence of the sphere. It is useful to obtain G^P_{pq} both for the sake of comparison and for the purpose of evaluating (124) in more detail.

As in Section IV, quantities in $\underset{\sim}{K}$ (transform) space will be labeled with a caret. Thus, the vectors $\hat{\underset{\sim}{\beta}}_i$ are defined as Fourier components of derivatives of the magnetic field even though the $\underset{\sim}{\beta}_i$ are defined only at the center of the sphere.

Employing (78) and changing the coordinates from ξ, η, ζ to x, y, z we have

$$B_{Pz} = \frac{K_z}{K_x} \hat{B}_{Px}. \tag{125}$$

Then, from (125) and the fact that

$$\nabla \cdot \underset{\sim}{B}_P = 0$$

it follows that

$$\hat{B}_{Px} = \frac{-i K_x}{K^2} \frac{\partial \hat{B}_{Py}}{\partial y}$$

and

$$\hat{B}_{Pz} = \frac{-i K_z}{K^2} \frac{\partial \hat{B}_{Py}}{\partial y} . \tag{126}$$

Now, let

$$\hat{\underset{\sim}{\beta}} = - \frac{\partial \hat{B}_{Py}}{\partial y} \frac{\underset{\sim}{K}}{K^2} - i \hat{B}_{Py} \underset{\sim}{i}_y . \tag{127}$$

Then

$$\hat{\underset{\sim}{\beta}}_1 = K_x \hat{\underset{\sim}{\beta}}, \quad \hat{\underset{\sim}{\beta}}_2 = K_z \underset{\sim}{\beta}, \quad \hat{\underset{\sim}{\beta}}_3 = i \frac{\partial \underset{\sim}{\beta}}{\partial y} . \tag{128}$$

From (78c) it can be seen that

$$\hat{B}_{Py} = \frac{\sigma\mu_o}{2K} \int_{-\infty}^{0} \left[-i \underset{\sim}{K} \cdot \underset{\sim}{B}_o + B_{oy} \frac{\partial}{\partial y} \right] e^{-K|y-y'|} \sum_n \phi_n(y') U_n dy' . \tag{129}$$

It follows from (127), (128), and (129) that

$$\hat{G}^{P}_{pq} = \underset{\sim}{1}_p \cdot \nabla(\underset{\sim}{1}_q \cdot \underset{\sim}{B}_p) = \frac{\sigma\mu_o}{2K} \int_{\infty}^{0} \left(T_1 + iT_2 \frac{\partial}{\partial y} + T_3 \frac{\partial^2}{\partial y^2} + iT_4 \frac{\partial}{\partial y^3} \right) e^{-K|y-y'|} \sum_n \phi_n(y') U_n dy', \tag{130}$$

where

$$T_1 = -1_{qy} (\underset{\sim}{K} \cdot \underset{\sim}{1}_p)(\underset{\sim}{K} \cdot \underset{\sim}{B}_o),$$

$$T_2 = -1_{py} 1_{qy} (\underset{\sim}{K} \cdot \underset{\sim}{B}_o) + \frac{(\underset{\sim}{K} \cdot \underset{\sim}{B}_o)(\underset{\sim}{K} \cdot \underset{\sim}{1}_p)(\underset{\sim}{K} \cdot \underset{\sim}{1}_q)}{K^2} - 1_{qy}(\underset{\sim}{K} \cdot \underset{\sim}{1}_p) B_{oy},$$

$$T_3 = 1_{py} 1_{qy} B_{oy} - \frac{1}{K^2} \left[1_{py} (\underset{\sim}{K} \cdot \underset{\sim}{1}_q)(\underset{\sim}{K} \cdot \underset{\sim}{B}_o) + (\underset{\sim}{K} \cdot \underset{\sim}{1}_p)(\underset{\sim}{K} \cdot \underset{\sim}{1}_q) B_{oy} \right],$$

$$T_4 = - \frac{1_{py} B_{oy} (\underset{\sim}{K} \cdot \underset{\sim}{1}_q)}{K^2}.$$

The derivative operations in (130) can be carried out with the aid of the following identities:

$$\begin{aligned} \frac{\partial}{\partial y} e^{-K|y-y'|} &= K[\eta(y'-y) - \eta(y-y')] e^{-K|y-y'|} \\ \frac{\partial^2}{\partial y^2} e^{-K|y-y'|} &= -2K\delta(y-y') + K^2 e^{-K|y-y'|}, \\ \frac{\partial^3}{\partial y^3} e^{-K|y-y'|} &= -2K\delta'(y-y') - K^3 e^{-K|y-y'|} \\ &\quad \cdot [\eta(y-y') - \eta(y'-y)], \end{aligned} \tag{131}$$

where $\eta(x)$ is the function defined by

$$\eta(x) = \begin{cases} 0, & y < 0 \\ 1, & y > 0 \end{cases}$$

and $\delta(x)$ is the Dirac delta function. With the aid of (131) equation (130) can be written

$$\hat{G}^{P}_{pq} = \frac{\sigma\mu_o}{2K}\left\{\left[T_1\int_{-\infty}^{0} + iKT_2\left(\int_{y}^{o} - \int_{-\infty}^{y}\right) + K^2T_3\int_{-\infty}^{0} + iK^3T_4\left(\int_{y}^{o} - \int_{-\infty}^{y}\right)\right] e^{-K|y-y'|}\right.$$

$$\times \sum_n \phi(y')\ U_n dy'$$

$$\left. -2KT_3 \sum_n \phi_n(y)U_n - 2iKT_4 \sum_n \dot{\phi}_n(y)U_n\right\}$$

$$= \frac{\sigma\mu_o}{2K}\left\{\left[\left(T_1 + K^2T_3\right)\int_{-\infty}^{0} + i(KT_2 + K^3T_4)\left(\int_{y}^{0} - \int_{-\infty}^{y}\right)\right] e^{-K|y-y'|}\right.$$

$$\times \sum_n \phi_n(y')U_n dy' \tag{132}$$

$$\left. -2KT_3 \sum_n \phi_n(y)U_n - 2iKT_4 \sum_n \dot{\phi}_n(y)U_n\right\}.$$

For the magnetic field gradient inside the sphere (122) can be used. The result will depend upon the moment vectors $\underset{\sim}{\mu}_i$ given by (97).

Using the coordinate system with the origin at the water surface rather than at the center of the sphere then (97) can be written

$$\underset{\sim}{\mu}_1 = \frac{3}{4\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\underset{\sim}{\omega}x'\ dx'dz'dy'}{[x'^2+z'^2+(y'-y)^2]^{3/2}},$$

$$\underset{\sim}{\mu}_2 = \frac{3}{4\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\underset{\sim}{\omega}z'dx'dz'dy'}{[x'^2+z'^2+(y'-y)^2]^{3/2}},$$

$$\underset{\sim}{\mu}_3 = \frac{3}{4\pi}\int_{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\underset{\sim}{\omega}(y'-y)dx'dz'dy'}{[x'^2+z'^2+(y'-y)^2]^{3/2}}\,, \tag{133}$$

where it is assumed that the center of the sphere is at the point with horizontal coordinates x = 0, z = 0, and the vertical coordinate y arbitrary. By considering the two dimensional Fourier transform with respect to x and z it can be shown that

$$\underset{\sim}{\mu}_1 = \frac{-3i}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{K_x}{K}\int_{-\infty}^{0} e^{-K|y'-y|}\,\hat{\underset{\sim}{\omega}}(\underset{\sim}{K},y')dy'd^2\underset{\sim}{K},$$

$$\underset{\sim}{\mu}_2 = \frac{-3i}{2}\int_{\infty}\int_{-\infty}^{\infty}\frac{K_z}{K}\int_{\infty}^{0} e^{-K|y'-y|}\,\hat{\underset{\sim}{\omega}}(\underset{\sim}{K},y')dy'd^2\underset{\sim}{K},$$

$$\underset{\sim}{\mu}_3 = \frac{3}{2}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{-\infty}^{0} e^{-K|y'-y|}\,[-\eta(y'-y) + \eta(y-y')]\,\hat{\underset{\sim}{\omega}}(\underset{\sim}{K},y')dy'd^2\underset{\sim}{K}$$

$$= -\frac{3}{2}\frac{\partial}{\partial y}\int_{\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\frac{e^{-K|y'-y|}}{K}\hat{\underset{\sim}{\omega}}(\underset{\sim}{K},y')dy'd^2\underset{\sim}{K}. \tag{134}$$

From (124), (127), (128), (129), (134), and (81) it is found that

$$\hat{G}_{pq} = \int_{-\infty}^{0}\left\{\frac{1}{2K}\left[L_1+L_2\frac{\partial}{\partial y} + L_3\frac{\partial^2}{\partial y^2} + L_4\frac{\partial^3}{\partial y^3}\right]e^{-K|y'-y|}\sum_n \phi_n(y')U_n\right.$$

$$\left. + \left[M_1+M_2\frac{\partial}{\partial y}\right]e^{-K|y'-y|}\,N^2(y')\sum_n\frac{\phi_n(y')}{\Omega_n^2}U_n\right\}dy', \tag{135}$$

where

$$L_1 = \frac{\sigma\mu_o}{2}(\underset{\sim}{K}\cdot\underset{\sim}{B}_o)(K_x\Lambda_{1y}+K_z\Lambda_{2y}),$$

$$L_2 = -\frac{i\sigma\mu_o}{2K^2}[K_x(\underset{\sim}{\Lambda}_1 \cdot \underset{\sim}{K})+K_z(\underset{\sim}{\Lambda}_2 \cdot \underset{\sim}{K})](\underset{\sim}{K} \cdot \underset{\sim}{B}_o) - \frac{i\sigma\mu_o}{2}\left[\left(K_x\Lambda_{1y}+K_z\Lambda_{2y}\right)B_{oy} + (\underset{\sim}{K} \cdot \underset{\sim}{B}_o)\Lambda_{3y}\right],$$

$$L_3 = -\frac{\sigma\mu_o}{2K^2}\left\{(\underset{\sim}{\Lambda}_3 \cdot \underset{\sim}{K})(\underset{\sim}{B}_o \cdot \underset{\sim}{K})+[K_x(\underset{\sim}{\Lambda}_1 \cdot \underset{\sim}{K})+K_z(\underset{\sim}{\Lambda}_2 \cdot \underset{\sim}{K})]B_{oy}\right\} - \frac{\sigma\mu_o}{2}\Lambda_{3y}B_{oy},$$

$$L_4 = -\frac{i\sigma\mu_o}{2K^2}(\underset{\sim}{\Lambda}_3 \cdot \underset{\sim}{K})B_{oy},$$

$$M_1 = \frac{\sigma\mu_o}{25K}[(K_z\Gamma_{1x}-K_x\Gamma_{1z})K_x+(K_z\Gamma_{2x}-K_x\Gamma_{2z})K_z],$$

$$M_2 = i\frac{\sigma\mu_o}{25K}(-K_z\Gamma_{3x}+K_x\Gamma_{3z}). \qquad (136)$$

The p,q dependence of $\hat{G}_{pq}$ is determined by the p,q dependence of the $\underset{\sim}{\Lambda}_i$ and $\underset{\sim}{\Gamma}_i$, which are defined in connection with (124). Thus, the L_i and the M_i depend upon p and q, although, to save space, these indices are not exhibited explicitly here. With the aid of (131) $\hat{G}_{pq}$ can also be written

$$\begin{aligned}\hat{G}_{pq} = &\int_{-\infty}^{o} \frac{1}{2K}(L_1 + K^2L_3)e^{-K|y'-y|}\sum_n \phi_n(y')U_n dy' \\ &+\int_{-\infty}^{o} M_1 e^{-K|y'-y|}N^2(y')\sum_n \frac{\phi_n(y')}{\Omega_n^2}U_n dy' \\ &+\int_y^o - \int_{-\infty}^{y} \frac{1}{2}(L_2 + K^2L_4)e^{-K|y'-y|}\sum_n \phi_n(y')U_n dy' \\ &+\int_y^o - \int_{-\infty}^{y} \frac{K}{2}M_2 e^{-K|y'-y|}N^2(y')\sum_n \frac{\phi_n(y')U_n dy'}{\Omega_n^2} \\ &-L_3\sum_n \phi_n(y)U_n - L_4\sum_n \dot{\phi}_n(y)U_n. \end{aligned} \qquad (137)$$

VII. SPECTRA OF MAGNETIC FIELD GRADIENTS

In order to obtain correlation functions $<\hat{G}_{pq}\ \hat{G}_{rs}>$, which are used to determine the spectral characteristics of the magnetic field gradient, it is convenient to write (135) in the form

$$\hat{G}_{pq} = \int_{-\infty}^{0} [A_{pq}(y,y')\sum_n \phi_n(y')U_n + B_{pq}(y,y')\sum_n \frac{\phi_n(y')}{\Omega_n^2} U_n]dy', \qquad (138)$$

where

$$A_{pq}(y,y') = \frac{1}{2K}\left[L_1 + L_2\frac{\partial}{\partial y} + L_3\frac{\partial^2}{\partial y^2} + L_4\frac{\partial^3}{\partial y^3}\right]e^{-K|y'-y|} \qquad (139)$$

and

$$B_{pq}(y,y') = [M_1 + M_2\frac{\partial}{\partial y}]\ e^{-K|y'-y|}N^2(y').$$

In (139) the quantities L_i and M_i depend upon the indices p,q.

With the aid of identities derived in Appendix E of Ref. 1, it can be seen that the correlation averages of magnetic field gradient components displaced in time are given by

$$<\hat{G}_{pq}(t)\hat{G}_{rs}(t')> = \int_{-\infty}^{0}\int_{-\infty}^{0}\left\{\left[A_{pq}(y,y')A^*_{rs}(y,y'')\right]\sum_n \phi_n(y')\phi_n(y'')V_n\right.$$
$$+\left[A_{pq}(y,y')B^*_{rs}(y,y'')+A^*_{rs}(y,y'')B_{pq}(y,y')\right]\sum_n \frac{\phi_n(y')\phi_n(y'')V_n}{\Omega_n^2}$$
$$\left.+\left[B_{pq}(y,y')B^*_{rs}(y,y'')\right]\sum_n \frac{\phi_n(y')\phi_n(y'')}{\Omega_n^4}V_n\right\} dy'dy'', \qquad (140)$$

where

$$V_n = \frac{1}{2}\left[\psi_n(\underset{\sim}{K})e^{i\Omega_n(K)\tau} + \psi_n(-\underset{\sim}{K})\, e^{-i\Omega_n(K)\tau}\right]$$

and

$$\tau = t'-t.$$

If the Milder hypothesis [1] is used in (140), this becomes

$$<\hat{G}_{pq}(t)\hat{G}_{rs}(t')> = \int_{-\infty}^{0}\int_{-\infty}^{0} A_{pq}(y,y')A^*_{rs}(y,y'')\sum_n \Omega_n^4(K)W_n(K,\tau)\phi_n(y')\phi_n(y'')dy'dy''$$

$$+\int_{-\infty}^{0}\int_{-\infty}^{0}\left[A_{pq}(y,y')B^*_{rs}(y,y'')+A^*_{rs}(y,y'')B_{pq}(y,y')\right]$$

$$\times \sum_n \Omega_n^2\,(K)W_n(\underset{\sim}{K},\tau)\phi_n(y')\phi_n(y'')dy'dy''$$

$$+\int_{-\infty}^{0}\int_{-\infty}^{0} B_{pq}(y,y')B^*_{rs}(y,y'')\sum_n W_n(\underset{\sim}{K},\tau)\phi_n(y')\phi_n(y'')dy'dy'', \tag{141}$$

where

$$W_n(\underset{\sim}{K},\tau) = \frac{1}{2\pi K^3}\left[I(\underset{\sim}{K})e^{i\Omega_n(K)\tau} + I(-\underset{\sim}{K})e^{-i\Omega_n(K)\tau}\right]. \tag{142}$$

Now, define

$$\gamma_\nu(y',y'') = \sum_n \Omega_n^\nu\, W_n\phi_n(y')\phi_n(y''),$$

$$A_{rs}^{(\nu)}(y,y') = \int_{-\infty}^{0} A^*_{rs}(y,y'')\ \gamma_\nu(y',y'')dy'', \tag{143}$$

and

$$B_{rs}^{(\nu)}(y,y') = \int_{-\infty}^{0} B^*_{rs}(y,y'')\ \gamma_\nu(y',y'')dy''.$$

Then, according to (141)

$$\langle \hat{G}_{pq}(t)\hat{G}_{rs}(t')\rangle = \int_{-\infty}^{o}\Big\{A_{pq}(y,y')\Big[A_{rs}^{(4)}(y,y') + B_{rs}^{(2)}(y,y')\Big] \tag{144}$$

$$+B_{pq}(y,y')\Big[A_{rs}^{(2)}(y,y') + B_{rs}^{(o)}(y,y')\Big]\Big\}\, dy'.$$

With the aid of (143), (131), and (139) it will be found that

$$A_{rs}^{(\nu)}(y,y') = \left(\frac{L_{1rs}}{2K} + \frac{KL_{3rs}}{2}\right)\int_{-\infty}^{o} e^{-K|y''-y|}\gamma_\nu(y',y'')dy'' \tag{145}$$

$$+\left(\frac{L^*_{2rs}}{2} + \frac{K^2L^*_{4rs}}{2}\right)\int_{y}^{o}-\int_{-\infty}^{y} e^{-K|y''-y|}\gamma_\nu(y',y'')dy''$$

$$-L_{3rs}\gamma_\nu(y',y) - L^*_{4rs}\gamma_{\nu y}(y',y)$$

and

$$B_{rs}^{(\nu)}(y,y') = \frac{M_{1rs}}{2K}\int_{-\infty}^{o} N^2(y'')e^{-K|y''-y|}\gamma_\nu(y',y'')dy''$$

$$+ \frac{M_{2rs}}{2}\int_{y}^{o}-\int_{-\infty}^{y} N^2(y'')e^{-K|y''-y|}\gamma_\nu(y',y'')dy''.$$

In (145) the subscript y on the last term of the expression for $A_{rs}^{(\nu)}(y,y')$ indicates, as usual, differentiation with respect to to the second argument. By similar means the relation (144) can be written

$$\langle \hat{G}_{pq}(t)\hat{G}_{rs}(t')\rangle = \left(\frac{L_{1pq}}{2} + \frac{KL_{3pq}}{2}\right)\int_{-\infty}^{0} e^{-K|y'-y|}\Big[A_{rs}^{(4)}(y,y') + B_{rs}^{(2)}(y,y')\Big]dy'$$

$$- \frac{M_{1pq}}{2K}\int_{-\infty}^{0} e^{-K|y'-y|}N^2(y')\Big[A_{rs}^{(2)}(y,y')+B_{rs}^{(o)}(y,y')\Big]dy'$$

$$+\left(\frac{L_{2pq}}{2}+\frac{K^2L_{4pq}}{2}\right)\int_y^0-\int_{-\infty}^y e^{-K|y'-y|}\left[A_{rs}^{(4)}(y,y') + B_{rs}^{(2)}(y,y')\right]dy'$$

$$-\frac{M_{2pq}}{2}\int_y^0-\int_{-\infty}^y e^{-K|y'-y|}N^2(y')\left[A_{rs}^{(2)}(y,y')+B_{rs}^{(o)}(y,y')\right]dy'$$

$$-L_{3pq}\left[A_{rs}^{(4)}(y,y)+B_{rs}^{(2)}(y,y)\right]-L_{4pq}\left[A_{rsy'}^{(4)}(y,y)+B_{rsy'}^{(2)}(y,y)\right]. \quad (146)$$

Again, the subscript y' in the last term of (146) indicates differentiation with respect to the second argument.

By comparing (130) with (135) and using (145) and (146) an expression like (146) can be found for $\langle \hat{G}^P_{pq}(t)\ \hat{G}^P_{rs}(t')\rangle$. The result is, in fact, (145) and (146) with M_1 and M_2 set equal to zero, the quantities L_1 and L_3 replaced by $\sigma\mu_o T_1$ and $\sigma\mu_o T_3$, and the quantities L_2 and L_4 replaced by $i\sigma\mu_o T_2$ and $i\sigma\mu_o T_4$. In this replacement the complex conjugates in (145), indicated by asterisks, are obviously obtained by simply changing sign, since the T_i are all real.

Since the statistical process is assumed to be stationary $\langle\hat{G}_{pq}(t)\hat{G}_{rs}(t')\rangle$ is actually a function of $\tau = t'-t$ in its time dependence. If $\langle\hat{G}_{pq}(t)\hat{G}_{rs}(t')\rangle$ is integrated over the two dimensional $\underset{\sim}{K}$ space and the temporal Fourier transform, with respect to τ, is taken, the result is the temporal spectral function $\Phi_{pq;rs}(\omega,y)$.

If this process is applied to (146) in two steps then the first step, integrating over $\underset{\sim}{K}$ space, leads to a sum of terms $I_\nu(\tau,y)$ of the form

$$I_\nu(\tau,y) = \int_{-\infty}^0 dy' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\underset{\sim}{K} \int_{-\infty}^0 dy''\ H_\nu(\underset{\sim}{K},\ y,\ y',\ y'')\gamma_\nu(y',y'') \quad (147)$$

The second step, taking the Fourier transform with respect to τ, proceeds as in Appendix E of Ref. 1.

The integration over $\underset{\sim}{K}$ space is done in polar coordinates after the temporal Fourier transform is taken. That is, the integral is taken over the magnitude K and the direction angle w of $\underset{\sim}{K}$. The integral over K is carried out explicitly and the angular integral is over the interval $(0, \frac{\pi}{2})$ after the change of variable

$$w = \alpha + \beta - \pi$$

from w to β.

When this procedure is applied to (147) and the derivation of equation (259) in Ref. 1 is imitated, an expression for the magnetic field gradient spectrum analogous to (259) in Ref. 1 will result. In carrying out the necessary steps the large tow speed approximation introduced in Ref. 1 is essential.

First, it is useful to consider the integrals $\tilde{J}_\nu(w,K,y)$, defined by

$$\tilde{J}_\nu(w,K,y) = \int_{-\infty}^{0}\int_{-\infty}^{0} H_\nu(\underset{\sim}{K},y,y',y'')\sum_n \Omega_n^\nu \phi_n(y')\phi_n(y'')dy'dy'', \qquad (148)$$

in more detail.

It is clear from (146) that the possible values of ν in (148) are 0, 2 and 4. Identities (F-9), (F-5), and (F-6) of Appendix F in Ref. 1 can be used to evaluate the series

$$S_\nu(y',y'') = \sum_n \Omega_n^\nu \phi_n(y')\phi_n(y'') \qquad (149)$$

for these values of ν.

For the case $\nu = 0$ the expression is simplified somewhat because

$$S_o(y',y'') = \sum_n \phi_n(y')\phi_n(y'') = \frac{\delta(y'-y'')}{N^2(y')} .$$

Thus,

$$\tilde{J}_o(w,K,y) = \int_{-\infty}^{0} \frac{1}{N^2(y')} H_o(\underset{\sim}{K},y,y',y')dy'. \tag{150}$$

For the other values of ν the calculation is more complicated; (148) becomes for $\nu = 2,4$:

$$\tilde{J}_\nu(w,K,y) = \int_{-\infty}^{0}\int_{-\infty}^{0} H_\nu(\underset{\sim}{K},y,y',y'')S_\nu(y',y'')dy'dy'', \tag{151}$$

where, according to (F-5), (F-6), and (F-8) of Ref. 1,

$$S_2(y',y'') = -K^2 g(y',y'')$$

and

$$S_4(y',y'') = K^4\int_{-\infty}^{0} N^2(x)g(x,y'')g(y',x)dx, \tag{152}$$

for g(u,v) defined by

$$g(u,v) = \frac{e^{Ku_<}\sinh Ku_>}{K}. \tag{153}$$

Inspection of (143) and (144) shows that

$$\begin{aligned} H_o(\underset{\sim}{K},y,y',y') &= B_{pq}(y,y')B^*_{rs}(y,y'), \\ H_2(\underset{\sim}{K},y,y',y'') &= A_{pq}(y,y')B^*_{rs}(y,y'')+B_{pq}(y,y')A^*_{rs}(y,y''), \\ H_4(\underset{\sim}{K},y,y',y'') &= A_{pq}(y,y')A^*_{rs}(y,y''). \end{aligned} \tag{154}$$

According to (131) and (139),

$$A(y,y') = \frac{1}{2K}\left\{ (L_1+K^2L_3) + K(L_2+K^2L_4)[\eta(y'-y)-\eta(y-y')]\right\} e^{-K|y-y'|}$$

$$-L_3\,\delta(y-y') - L_4\,\delta'(y-y') \tag{155}$$

and

$$B(y,y') = N^2(y')\left\{M_1 + KM_2[\eta(y'-y)-\eta(y-y')]\right\}e,^{-K|y-y'|}$$

where subscripts p, q and r, s are not shown but are understood to be determined by the constants L_1 and M_1.

It follows from (150), (151), and the derivation given for the relation (259) of Ref. 1 that the temporal spectrum of the magnetic field gradient is given by

$$\Phi_{pq;rs}(\omega,y) = \frac{1}{\omega}\int_0^{\frac{\pi}{2}} \frac{1}{K}\left\{\tilde{J}_{pq;rs}(\alpha+\beta-\pi,K)[I(K,\alpha+\beta-\pi)+I(K,\alpha+\beta)]\right. \tag{156}$$

$$\left. + \tilde{J}_{pq;rs}(\alpha-\beta-\pi,K)[I(K,\alpha-\beta-\pi)+I(K,\alpha-\beta)]\right\} d\beta,$$

where

$$\tilde{J}(w,K) = \tilde{J}_o(w,K) + \tilde{J}_2(w,K) + \tilde{J}_4(w,K). \tag{157}$$

The magnetic field gradient temporal spectrum in the absence of the bubble can be obtained from (156) by setting the M_1 equal to zero and replacing L_1 by $\sigma\mu_o T_1$, L_2 by $i\sigma\mu_o T_2$, L_3 by $\sigma\mu_o T_3$, and L_4 by $i\sigma\mu_o T_4$ in (155). The result is

$$\Phi^P_{pq;rs}(\omega,y) = \frac{1}{\omega}\int_0^{\frac{\pi}{2}} \frac{1}{K}\left\{\tilde{J}^P_{pq;rs}(\alpha+\beta-\pi,K,y)[I(K,\alpha+\beta-\pi)+I(K,\alpha+\beta)]\right. \tag{158}$$

$$\left. + \tilde{J}^P_{pq;rs}(\alpha-\beta-\pi,K,y)[I(K,\alpha-\beta-\pi)+I(K,\alpha-\beta)]\right\} d\beta,$$

where

$$\tilde{J}^P_{pq;rs} = \int_{-\infty}^{o}\int_{-\infty}^{o} A^P_{pq}(y,y')A^{P*}_{rs}(y,y")S_4(y',y")dy'dy"$$

and where

$$A^P(y,y') = \frac{\sigma\mu_o}{2K}\left\{T_1 + K^2T_3 + iK(T_2+K^2T_4)[\eta(y'-y)-\eta(y-y')]\right\}e^{-K|y-y'|}$$

$$- \sigma\mu_o\left\{T_3\delta(y-y') + iT_4\delta'(y-y')\right\}.$$

Each of the terms in (156) can be reduced, so that the magnetic field gradient $\Phi_{pq;rs}(\omega,y)$ can be expressed as an integral over β of a quantity that involves just one more quadrature. For this purpose it is convenient to define functions $f_\nu(K,y)$ by means of this quadrature:

$$f_\nu(K,y) = \int_y^0 x^\nu e^{Kx} N^2(x)dx. \tag{159}$$

The first term in (157) is the quantity $\tilde{J}_o(w,K,y)$ given by (150). The relations (154) and (155) are needed in evaluating that term, as well as the others. It is found that

$$\int_{-\infty}^0 \frac{1}{N^2(y')} H_o(\underset{\sim}{K},y,y',y')dy' = C_{o1}\int_{-\infty}^0 N^2(y')e^{-2K|y-y'|}dy'$$

$$+ C_{o2}\left(\int_y^0 - \int_{-\infty}^y\right) N^2(y')e^{-2K|y-y'|}dy'$$

$$= (C_{o1}+C_{o2})e^{2Ky}\int_y^0 e^{-2Ky'}N^2(y')dy' \tag{A-2}$$

$$+ (C_{o1}-C_{o2})e^{-2Ky}\int_{-\infty}^y e^{2Ky'}N^2(y')dy'$$

$$= (C_{o1}+C_{o2})e^{2Ky}f_o(-2K,y)+(C_{o1}-C_{o2})e^{-2Ky}[f_o(2K,-\infty)-f_o(2K,y)], \tag{160}$$

where

$$C_{o1} = \frac{1}{4}\left(\frac{1}{K^2} M_{1pq}M^*_{1rs} + M_{2pq}M^*_{2rs}\right) \tag{161}$$

and

$$C_{o2} = \frac{1}{4K}(M_{1pq}M^*_{2rs}+M_{2pq}M^*_{1rs}).$$

For evaluating the second term in (157) it is convenient to define a function $F(y',y)$ by

$$F(y',y) = \int_{-\infty}^{0} S_2(y',y'')\ A(y,y'')dy''. \tag{162}$$

The quantities appearing in (162) are given by (152), (153), and (155).

Then, because of (154), it is found that

$$\int_{-\infty}^{0}\int_{-\infty}^{0} H_2(\underset{\sim}{K},y,y',y'')S_2(y',y'')dy'dy'' = \int_{-\infty}^{0} [B_{pq}(y,y')F^*_{rs}(y',y)+B^*_{rs}(y,y')F_{pq}(y',y)]dy'. \tag{163}$$

The symmetry of the function $S_2(y',y'')$ is used in deriving (162).

Further evaluation of the second term in (157) proceeds from (162) by means of the equation

$$-\int_{-\infty}^{0} B(y,y')F(y',y)dy' = \frac{1}{2}\left(\frac{M_1}{K} + M_2\right)e^{-Ky}\int_{-\infty}^{y} e^{Ky'}N^2(y')F(y',y)dy'$$

$$+ \frac{1}{2}\left(\frac{M_1}{K} - M_2\right)e^{Ky}\int_{y}^{0} e^{-Ky'}N^2(y')F(y',y)dy'. \tag{164}$$

The subscripts p, q, r, s and the complex conjugates in (163) can be added to the general relation (164) by using the appropriate subscripts and complex conjugates for the quantities L_1, L_2, L_3, L_4, M_1, and M_2, which are all independent of the integration variables in (162) and (163).

The evaluation is continued by proceeding from (162). Thus, using notation the meaning of which should be obvious,

$$F(y'<y) = -K[-(L_3 \sinh Ky + L_4 \cosh Ky)e^{Ky'}+(C_{21}-C_{22})e^{-Ky}\sinh Ky'\int_{-\infty}^{y'} e^{2Ky''}dy''$$

$$+(C_{21}-C_{22})e^{K(y'-y)}\int_{y'}^{y} e^{Ky''}\sinh Ky''\ dy''$$

$$+(C_{21}+C_{22})e^{K(y'+y)}\int_y^o e^{-Ky''}\sinh Ky''\, dy'']$$

$$= -K\Big\{-(L_3 \sinh Ky + KL_4 \cosh Ky)e^{Ky'} + \frac{1}{2K}(C_{21}-C_{22})e^{-Ky}e^{2Ky'} \sinh Ky'$$

$$+ \frac{1}{2}(C_{21}-C_{22})[\frac{1}{2K}(e^{2Ky}-e^{2Ky'}) + y'-y]e^{K(y'-y)} + \frac{1}{2}(C_{21}+C_{22})e^{K(y+y')}$$

$$x\, [-y + \frac{1}{2K}(1-e^{-2Ky})]\Big\}$$

$$= F_o(y)e^{Ky'} + F_1(y)y'e^{Ky'}, \tag{165}$$

where

$$F_o(y) = K[(L_3 - \frac{C_{21}}{K})\sinh Ky + KL_4 \cosh Ky + \frac{1}{2}(C_{21}+C_{22})ye^{Ky}$$

$$+ \frac{1}{2}(C_{21}-C_{22})ye^{-Ky}, \tag{166}$$

$$F_1(y) = -\frac{1}{2}K(C_{21}-C_{22})e^{-Ky}$$

$$C_{21} = \frac{1}{2K}(L_1 + K^2L_3) \tag{167}$$

and

$$C_{22} = \frac{1}{2}(L_2 + K^2L_4)$$

Similarly,

$$F(y' > y) = -K\Big\{-(L_3+KL_4)e^{Ky}\sinh Ky' + \sinh Ky'\, [(C_{21}-C_{22})e^{-Ky}\int_{-\infty}^{y} e^{2Ky''}dy''$$

$$+ (C_{21}+C_{22})e^{KY}\int_y^{y'} dy''] + (C_{21}+C_{22})e^{K(y'+y)}\int_{y'}^{o} e^{-Ky''}\sinh Ky''\, dy''\Big\}$$

$$= -K\Big\{-(L_3+KL_4)e^{Ky} \sinh Ky' + e^{Ky} \sinh Ky'\, [\frac{1}{2K}(C_{21}-C_{22})+(C_{21}+C_{22})(y'-y)]$$

$$+ \frac{1}{2}(C_{21}+C_{22})e^{K(y'+y)}[-y' + \frac{1}{2K}(1-e^{-2Ky'})]\Big\} \tag{168}$$

$$= F_2(y)e^{Ky'}-F_2(y)e^{-Ky'}+F_3(y)y'e^{-Ky'},$$

where

$$F_2(y) = \frac{K}{2}\left[L_3 + KL_4 - \frac{1}{K}C_{21} + (C_{21} + C_{22})y\right]e^{Ky} \tag{169}$$

and

$$F_3(y) = \frac{K}{2}(C_{21} + C_{22})e^{Ky}.$$

Inserting (165) and (168) into (164)

$$\int_{-\infty}^{0} B(y,y')F(y',y)dy' = \frac{1}{2}\left(\frac{M_1}{K} + M_2\right)e^{-Ky}\Big\{F_o(y)[f_o(2K,-\infty)-f_o(2K,y)] + F_1(y)[f_1(2K,-\infty)-f_1(2K,y)]\Big\} + \frac{1}{2}\left(\frac{M_1}{K} - M_2\right)e^{-Ky}\Big\{F_2(y)[f_o(o,y)-f_o(-2K,y)]+F_3(y)f_1(-2K,y)\Big\}. \tag{170}$$

The desired expression (163) is obtained from two applications of (170).

The subscripts on B in (163), as well as the complex conjugation, are obtained by applying the same subscripts, and complex conjugation, to the quantities M_1 and M_2 appearing explicitly in (170). The same rule applies to A with subscripts and complex conjugation applied to L_3, L_4, C_{21}, and C_{22} appearing explicitly in (166) and (169).

In the last term of (157) the expression to be evaluated is, because of (151), (152), and (154),

$$\int_{-\infty}^{0}\int_{-\infty}^{0} H_4(\underset{\sim}{K},y,y',y'')S_4(y',y'')dy'' = K^4\int_{-\infty}^{0} N^2(x)G_{pq}(y,x)G^{*}_{rs}(y,x)dx, \tag{171}$$

where

$$G(y,x) = \int_{-\infty}^{0} A(y,y')g(y',x)dy'. \tag{172}$$

In (171) the subscripts on G(y,x) and the complex conjugation actually occur through A(y,y') in (172).

With the aid of (155) the expression (172) can be written

$$G(y,x) = -L_3 g(y,x) - L_4 g_y(y,x) \tag{173}$$

$$+(C_{21}-C_{22})e^{-Ky}\int_{-\infty}^{y} e^{Ky'} g(y',x)dy' + (C_{21}+C_{22})e^{Ky}\int_{y}^{0} e^{-Ky'} g(y',x)dy'.$$

From (153) it follows that

$$G(y > x) = -\left(\frac{L_3}{K}\sinh Ky + L_4 \cosh Ky\right)e^{Kx} + \frac{1}{K}(C_{21}-C_{22})e^{-Ky}\left[\sinh Kx\int_{-\infty}^{x} e^{2Ky'}dy' + e^{Kx}\int_{x}^{y} e^{Ky'} \sinh Ky'\, dy'\right] \tag{174}$$

$$+ \frac{1}{K}(A_1 + A_2)\, e^{K(y+x)}\int_{y}^{0} e^{-Ky'}\sinh Ky' dy'$$

$$= G_o(y)e^{Kx} + G_1(y)xe^{Kx},$$

where

$$G_o(y) = \frac{1}{K}(C_{21}+C_{22})\left(\frac{\sinh Ky}{K} - ye^{Ky}\right) + \frac{1}{2K}(C_{21}-C_{22})\left(\frac{\sinh Ky}{K} - ye^{-Ky}\right)$$

$$-L_3\frac{\sinh Ky}{K} - L_4 \cosh Ky \tag{175}$$

and

$$G_1(y) = \frac{1}{2K}(C_{21}-C_{22})e^{-Ky}.$$

It also follows that

$$G(y<x) = -\left(\frac{L_3}{K} + L_4\right)e^{Ky}\sinh Kx + \frac{1}{K}(C_{21}-C_{22})e^{-Ky}\sinh Kx\int_{-\infty}^{y} e^{2Ky'}dy'$$

$$+ \frac{1}{K}(C_{21}+C_{22})e^{Ky}\left[\sinh Kx\int_{y}^{x} dy' + e^{Kx}\int_{x}^{0} e^{-Ky'}\sinh Ky'dy'\right] \tag{176}$$

$$= G_2(y)e^{Kx} - G_2(y)e^{-Kx} + G_3(y)xe^{-Kx}, \tag{177}$$

where

$$G_2(y) = \frac{1}{2}\ \left[\frac{C_{21}}{K^2} - \frac{L_3}{K} - L_4 - \frac{1}{K}(C_{21}+C_{22})y\right]e^{Ky}$$

and

$$G_3(y) = -\frac{1}{2K}(C_{21}+C_{22})e^{Ky}.$$

It follows from (171), (174), and (176) that

$$\int_{-\infty}^{o}\int_{-\infty}^{o} H_4(\underset{\sim}{K},y,y',y")S_4(y',y")dy'dy"$$

$$= K^4\int_{-\infty}^{y} N^2(x)[G_{opq}(y)e^{Kx}+G_{1pq}(y)xe^{Kx}][G^*_{ors}(y)e^{Kx}+G^*_{1rs}(y)xe^{Kx}]dx$$

$$+ K^4\int_{y}^{o} N^2(x)[G_{2pq}(y)(e^{Kx}-e^{-Kx})+G_{3pq}(y)xe^{Kx}][G^*_{2rs}(y)\left(e^{Kx}-e^{-Kx}\right)+G^*_{3rs}(y)xe^{Kx}]dx$$

$$= K^4\Big\{ H_o(y)[f_o(2K,-\infty)-f_o(2K,y)]+H_1(y)[f_1(2K,-\infty)-f_1(2K,y)]$$

$$+H_2(y)[f_2(2K,-\infty)-f_2(2K,y)]+H_{o-1}(y)f_o(-2K,y)+H_{oo}(y)f_o(o,y)$$

$$+H_{o1}(y)f_o(2K,y)+H_{1o}(y)f_1(o,y)+H_{11}(y)f_1(2K,y)+H_{21}(y)f_2(2K,y)\Big\}, \quad (178)$$

where

$$H_o(y)=G_{opq}(y)G^*_{ors}(y), H_1(y) = G_{1pq}(y)G^*_{ors}(y)+G_{opq}(y)G^*_{1rs}(y),$$

$$H_2(y)=G_{1pq}(y)G^*_{1rs}(y), H_{o-1}(y)=G_{2pq}(y)G^*_{2rs}(y), H_{oo}(y)= -2G_{2pq}(y)G^*_{2rs}(y),$$

$$H_{o1}(y)=G_{2pq}(y)G^*_{2rs}(y), H_{1o}(y) = -G_{2pq}(y)G^*_{3rs}(y)-G_{3pq}(y)G^*_{2rs}(y),$$

$$H_{11}(y)=G_{2pq}(y)G^*_{3rs}(y)+G_{3pq}(y)G^*_{2rs}(y), H_{21}(y) = G_{3pq}(y)G^*_{3rs}(y). \quad (179)$$

When there is no sphere, the magnetic field gradient is that due to internal waves for the case in Ref. 1 where the field is observed below the sea surface. When the gradient contains a vertical (y) derivative it will have a discontinuity

at the sea surface, but if the derivative is in the horizontal (x or z) direction it should be continuous at the surface. Thus, the quantity $\Phi_{13;13}$ given here for the case of no sphere should agree with (259) in Ref. 1 when both are evaluated at y = 0 if the assumptions that were made in Ref. 1 in deriving (259) are also made here. These assumptions include the Milder hypothesis and a large tow speed, both of which have already been adopted here. In addition the assumption of an isotropic internal wave excitation function with power law dependence on wave number was also used in Ref. 1. That is, it was assumed that

$$I(\underset{\sim}{K}) = CK^{-p}$$

For the case of no sphere, as already observed in connection with (157), $\tilde{J}$ reduces to $\tilde{J}_4$, and at y = 0, according to (175), $\tilde{J}_4$ reduces to

$$\begin{aligned}\tilde{J}^{P}_{pq;r,s}(w,K,o) &= K^4 G_{1pq}(o)G^*_{1rs}f_2(2K,-\infty)\\ &= \frac{K^2}{4}(C_{21pq}-C_{22pq})(C^*_{21rs}-C^*_{22rs})f_2(2K,-\infty)\\ &= \frac{K^2\sigma^2\mu_o^2}{16}\left(\frac{T_{1pq}}{K}+KT_{3pq}-iT_{2pq}\right)\left(\frac{T_{1rs}}{K}+KT_{3rs}+iT_{2rs}\right)f_2(2K,-\infty).\end{aligned} \tag{180}$$

For the last expression in (180) the rules for substituting T_i in the case of no bubble were used in (167).

According to the definition of the T_i in connection with (130) and the dependence, given by (113) in Ref. 1, of $\underset{\sim}{B}_o$ on ϕ_D, and by (120) in Ref. 1 for the $\underset{\sim}{1}_p$ vectors on α, it is a straightforward matter to verify that

$$T_{113} = o,\ T_{213} = B_o K\cos(w-\alpha)\sin(w-\alpha)\cos w \cos \phi_D, \tag{181}$$

$$T_{313} = -B_o \cos (w-\alpha) \sin (w-\alpha) \sin \phi_D.$$

If (181) is substituted into (180) it is found that

$$\tilde{J}^{P}_{13;13}(w,K,o) = \frac{2\sigma^2\mu_o^2 B_p^2 K^4}{(1+3\cos^2\phi_D)^2}\, g_{13;13}(w,\alpha) f_2(2K,-\infty), \tag{182}$$

where $g_{13;13}(w,\alpha)$ is defined by equation (182f) in Ref. 1.

Finally, if (182) is used in (158) along with (179) it will be found that the result agrees with (259) of Ref. 1. In order to complete the verification it is necessary to use (159) to identify $f_2(2K,-\infty)$.

APPENDIX A

PROOF OF THE DIRICHLET GREEN'S FUNCTION APPROXIMATION

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The notation to be used in the following is consistent with that of the preceding text. Namely, $G_S(\underset{\sim}{r},\underset{\sim}{r}')$ is the exterior Green's function for S (G_s vanishes on S) and $G_P(\underset{\sim}{r},\underset{\sim}{r}')$ is the Green's function for V_P (G_P vanishes on P). In fact, with $G_o(\underset{\sim}{r},\underset{\sim}{r}')$ the free space Green's function given by

$$G_o = \frac{1}{4\pi} \frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} ,$$

$$G_{DP}(\underset{\sim}{r},\underset{\sim}{r}') = G_o(\underset{\sim}{r},\underset{\sim}{r}') - G_o(\hat{\underset{\sim}{r}},\underset{\sim}{r}')$$

and

$$G_{DS}(\underset{\sim}{r},\underset{\sim}{r}') = G_o(\underset{\sim}{r},\underset{\sim}{r}') - \frac{a}{4\pi\sqrt{a^4 - 2a^2rr'\cos\gamma + r^2r'^2}} . \tag{A-1}$$

In addition, operators D_S and D_P are defined by

$$D_S G_1 = \int_S \frac{\partial G_S(\underset{\sim}{r},\underset{\sim}{r}'')}{\partial r''} G_1(\underset{\sim}{r}'',\underset{\sim}{r}')ds'' \tag{A-2}$$

and

$$D_P G_1 = \int_S \frac{\partial G_S(\hat{\underset{\sim}{r}},\underset{\sim}{r}'')}{\partial r''} G_1(\underset{\sim}{r}'',\underset{\sim}{r}')ds''. \tag{A-3}$$

According to Green's theorem (A-2) implies that $D_S G_1$ is equal to G_1 on S and satisfies Laplace's equation outside of S. According to (A-3) $D_P G_1$ is equal to $D_S G_1$ on P and satisfies Laplace's equation in V_{PS}.

Let

$$G_1(\underset{\sim}{r},\underset{\sim}{r}') = G_{DS}(\underset{\sim}{r},\underset{\sim}{r}') - G_{DS}(\hat{\underset{\sim}{r}},\underset{\sim}{r}'). \qquad \text{(A-4)}$$

Then G_1 vanishes on P but not on S. However, $G_1-D_SG_1$ vanishes on S but not on P. Further, $G_1-D_sG_1 + D_PG_1$ vanishes on P but not on S, and $G_1-D_SG_1 + D_PG_1-D_SD_PG_1$ vanishes on S but not on P. Continuing the iteration <u>ad infinitum</u> leads to a Green's function $G_D(\underset{\sim}{r},\underset{\sim}{r}')$ defined by

$$G_D(\underset{\sim}{r},\underset{\sim}{r}') = (1-D_S)\sum_{n=o}^{\infty} D_P{}^n \, G_D = (1-D_S)(1-D_P)^{-1} G_o, \qquad \text{(A-5)}$$

if the series converges. The limiting function vanishes on both S and P, as required for the Dirichlet Green's function appropriate to the region V_{PS} bounded by P and S.

Convergence of the series in (A-5) occurs in the sense of any norm for which

$$||D_P|| < 1. \qquad \text{(A-6)}$$

For any function $f(\underset{\sim}{r}")$

$$D_Pf = a^2\int_\Omega \frac{\partial G_S(\hat{\underset{\sim}{r}},\underset{\sim}{r}")}{\partial r"} f(\underset{\sim}{r}")\Big|_{r"=a} d\,\Omega$$

$$= \frac{a^2}{4\pi}\int_\Omega \left[\frac{-a+\hat{r}\cos\gamma}{(\hat{r}^2+a^2-2a\hat{r}\cos\gamma)^{3/2}} + \frac{\hat{r}^2-a\hat{r}\cos\gamma}{a(a^2-2a\hat{r}\cos\gamma+\hat{r}^2)^{3/2}}\right] f\Big|_{r"=a} d\Omega$$

$$= \frac{a}{4\pi\hat{r}}\int_\Omega \left[\frac{\left(\cos\gamma - \frac{a}{\hat{r}}\right)\frac{a}{\hat{r}}}{\left(1+\frac{a^2}{\hat{r}^2} - \frac{2a}{\hat{r}}\cos\gamma\right)^{3/2}} + \frac{1-\frac{a}{\hat{r}}\cos\gamma}{\left(1-\frac{2a}{\hat{r}}\cos\gamma+\frac{a^2}{\hat{r}^2}\right)^{3/2}}\right] f\Big|_{r"=a} d\Omega,$$

so that

$$||D_P\ f|| \le \frac{\frac{a}{D}\left(1 + \frac{a}{D}\right)}{(1-\frac{a}{D})^2}||f|| \tag{A-7}$$

if the norm of a function is the maximum value of the function on S. Convergence is then guaranteed as long as

$$\frac{a}{D} < \frac{1}{3}. \tag{A-8}$$

In forming the series that converges to $G(\underset{\sim}{r},\underset{\sim}{r}')$ each step adds a term one order higher in $\frac{a}{D}$. In fact, the series begins with G_o, the free space Green's function and $G_1 = (1-D_S)G_o$. Thus, the lowest order approximation to $G_D(\underset{\sim}{r},\underset{\sim}{r}')$ is $G_1(\underset{\sim}{r},\underset{\sim}{r}')$ which is also given by (A-4).

By definition,

$$\Delta G = G_D - G_{DP}.$$

Hence,

$$\Delta G \sim G_1 - G_{DP},$$

so that, according to (A-1) and (A-4), ΔG is given approximately by (60), as asserted earlier.

APPENDIX B

CONTRIBUTION OF THE PARTIAL ELECTRIC FIELD CURRENT TO THE MAGNETIC FIELD

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In the following it will be shown that the partial electric field contribution $\underset{\sim}{J}_E$ to the current given by (47) results in a contribution of higher order than a to the magnetic field inside the sphere. Then it will follow, once again, that only the term $\underset{\sim}{J}_M$ in (47) need be considered.

The potential ϕ' whose gradient determines $\underset{\sim}{J}_E$ satisfies Laplace's equation inside V_S and the boundary condition

$$\frac{\partial\phi'}{\partial r}\Big|_{r=a} = \underset{\sim}{i}_r \cdot (\underset{\sim}{B}_o \cdot \nabla)\underset{\sim}{\psi}\Big|_{r=a} \qquad \text{(B-1)}$$

on S. From (B-1), (53), and (64) it follows that, to terms of order less than a,

$$\frac{\partial\phi'}{\partial r}\Big|_{r=a} \sim \frac{(\underset{\sim}{i}_r \cdot \underset{\sim}{\mu})\,(\underset{\sim}{i}_r \cdot \underset{\sim}{B}_o)}{a} + \underset{\sim}{i}_r \cdot \underset{\sim}{P}, \qquad \text{(B-2)}$$

where

$$\underset{\sim}{P} = (\underset{\sim}{B}_o \cdot \nabla)\underset{\sim}{\psi}_P\Big|_{r=a}. \qquad \text{(B-3)}$$

In obtaining (B-2) terms of order $\frac{a}{D}$ have been dropped as usual.

Now, for any two constant vectors $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ the quantity $\sum_2$ given by

$$\sum\nolimits_2 = (\underset{\sim}{i}_r \cdot \underset{\sim}{k}_1)(i_r \cdot \underset{\sim}{k}_2) - \frac{1}{3}(\underset{\sim}{k}_1 \cdot \underset{\sim}{k}_2) \qquad \text{(B-4)}$$

is a surface spherical harmonic of the second degree [3]. Also, for any constant vector $\underset{\sim}{k}$ the quantity $\sum_1$ given by

$$\sum\nolimits_1 = \underset{\sim}{i}_r \cdot \underset{\sim}{k} \qquad \text{(B-5)}$$

is a surface spherical harmonic of degree one, and, of course, any constant $\sum_0$ is a surface spherical harmonic of degree zero.

It follows from (B-2), (B-4), and (B-5), for

$$Z_o(\phi,\theta) = \frac{(\underset{\sim}{\mu} \cdot B_o)}{3a},$$

$$Z_1(\phi,\theta) = \underset{\sim}{i}_r \cdot \underset{\sim}{P}, \tag{B-6}$$

and

$$Z_2(\phi,\theta) = \frac{1}{a}\left[\left(\underset{\sim}{i}_r \cdot \underset{\sim}{\mu}\right)\left(\underset{\sim}{i}_r \cdot \underset{\sim}{B}_o\right) - \frac{1}{3}\left(\underset{\sim}{\mu} \cdot \underset{\sim}{B}_o\right)\right],$$

that

$$\phi' \sim -\sum_{n=0}^{2} \frac{a^{n+2}}{(n+1)r^{n+1}} Z_n(\phi,\theta). \tag{B-7}$$

In (B-7) the quantities Z_n (ϕ,θ) are surface spherical harmonics of degree n.

From (B-7) it follows that

$$\nabla\phi' \sim \sum_{n=o}^{2}\left(\frac{a}{r}\right)^{n+2}\left[Z_n(\phi,\theta)\underset{\sim}{i}_r - \frac{\nabla_\Omega Z_n(\phi,\theta)}{n+1}\right],$$

so that

$$\underset{\sim}{i}_r \times \nabla\phi' \sim -\sum_{n=1}^{2}\left(\frac{a}{r}\right)^{n+2}\frac{\underset{\sim}{i}_r \times \nabla_\Omega Z_n(\phi,\theta)}{n+1}. \tag{B-8}$$

Thus,

$$\left[\underset{\sim}{i}_r \times \nabla\phi'\right]_{r=a} \sim \underset{\sim}{I}_1 + \underset{\sim}{I}_2, \tag{B-9}$$

where

$$\mathbf{I}_1 = -\frac{1}{2}\,\mathbf{i}_r \times \nabla_\Omega\,(\mathbf{i}_r \cdot \mathbf{P})$$

and

$$I_2 = -\mathbf{i}_r \times \nabla_\Omega\,\frac{[\mathbf{i}_r \cdot \boldsymbol{\mu})(\mathbf{i}_r \cdot \mathbf{B}_o)]}{3a}\,.$$

From

$$\nabla(\mathbf{r} \cdot \mathbf{P}) = \mathbf{P} = \nabla(r\mathbf{i}_r \cdot \mathbf{P}) = (\mathbf{i}_r \cdot \mathbf{P})\mathbf{i}_r + r\nabla(\mathbf{i}_r \cdot \mathbf{P})$$

$$= (\mathbf{i}_r \cdot \mathbf{P})\mathbf{i}_r + \nabla_\Omega\,(\mathbf{i}_r \cdot \mathbf{P})$$

it follows that

$$\mathbf{i}_r \times \nabla_\Omega\,(\mathbf{i}_r \cdot \mathbf{P}) = \mathbf{i}_r \times \mathbf{P}. \tag{B-10}$$

Since (B-10) is an identity that holds for an arbitrary constant vector $\mathbf{P}$ it can be applied to $\mathbf{I}_2$, with the result that

$$\mathbf{I}_2 = -\frac{1}{3a}\,\mathbf{i}_r \times [(\mathbf{i}_r \cdot \mathbf{B}_o)\nabla_\Omega(\mathbf{i}_r \cdot \boldsymbol{\mu}) + (\mathbf{i}_r \cdot \boldsymbol{\mu})\nabla_\Omega(\mathbf{i}_r \cdot \mathbf{B}_o)]$$

$$= -\frac{1}{3a}\,\mathbf{i}_r \times [(\mathbf{i}_r \cdot \mathbf{B}_o)\boldsymbol{\mu} + (\mathbf{i}_r \cdot \boldsymbol{\mu})\mathbf{B}_o]. \tag{B-11}$$

It is useful to define

$$\boldsymbol{\nu} = \mathbf{B}_o \times \boldsymbol{\mu},\ \boldsymbol{\lambda}_1 = \boldsymbol{\nu} \times \boldsymbol{\mu},\ \boldsymbol{\lambda}_2 = \boldsymbol{\nu} \times \mathbf{B}_o\ . \tag{B-12}$$

Then

$$\boldsymbol{\mu} = \frac{\boldsymbol{\lambda}_1 \times \boldsymbol{\nu}}{\nu^2}\ ,\ \mathbf{B}_o = \frac{\boldsymbol{\lambda}_2 \times \boldsymbol{\nu}}{\nu^2}\ . \tag{B-13}$$

From (B-11) and (B-12) it follows that

$$\mathbf{I}_2 = -\frac{1}{3a}\Bigg[\Big(\mathbf{i}_r \cdot \mathbf{B}_o\Big)\Big(\mathbf{i}_r \cdot \boldsymbol{\nu}\Big)\frac{\boldsymbol{\lambda}_1}{\nu^2}\ \Big(\mathbf{i}_r \cdot \mathbf{B}_o\Big)\Big(\mathbf{i}_r \cdot \boldsymbol{\lambda}_1\Big)\frac{\boldsymbol{\nu}}{\nu^2}$$

$$+ (\mathbf{i}_r \cdot \boldsymbol{\mu})(\mathbf{i}_r \cdot \boldsymbol{\nu})\,\frac{\boldsymbol{\lambda}_2}{\nu^2} - (\mathbf{i}_r \cdot \boldsymbol{\mu})(\mathbf{i}_r \cdot \boldsymbol{\lambda}_2)\,\frac{\boldsymbol{\nu}}{\nu^2}\Bigg]\ . \tag{B-14}$$

From the form (B-4) it can be seen that each cartesian component of $\underset{\sim}{I}_2$ is a sum of a spherical harmonic of degree two and a spherical harmonic of degree zero (constant). A simple calculation shows that the spherical harmonic of degree zero, in fact, vanishes. Thus, $\underset{\sim}{I}_2$ is a vector whose cartesian components are each spherical harmonics of degree two. Because of (B-10) it is apparent that $\underset{\sim}{I}_1$ is a vector whose cartesian components are each spherical harmonics of degree one.

Then, as indicated by (52), (51) becomes

$$\underset{\sim}{B}_E\Big|_{r=a} \equiv \underset{\sim}{I}_E \sim \sigma\mu_o \frac{a}{4\pi}\sum_{n=1}^{2}\int_{\Omega} P_n(\cos\gamma)\, \underset{\sim}{I}_n(\phi',\theta')d\Omega$$

$$= \sigma\mu_o a\left[\frac{\underset{\sim}{I}_1(\phi,\theta)}{3} + \frac{\underset{\sim}{I}_2(\phi,\theta)}{5}\right], \tag{B-15}$$

by virtue of an identity given in Ref. 3, p. 137. It follows from the definition of $\underset{\sim}{I}_1$ in (B-9), from (B-10), and from (B-11) that

$$\underset{\sim}{I}_E \sim \sigma\mu_o\left\{\frac{a}{6}\, \underset{\sim}{i}_r \times \underset{\sim}{P} + \frac{1}{15}\, \underset{\sim}{i}_r \times \left[(\underset{\sim}{i}_r \cdot \underset{\sim}{B}_o)\underset{\sim}{\mu} + (\underset{\sim}{i}_r \cdot \underset{\sim}{\mu})\underset{\sim}{B}_o\right]\right\}. \tag{B-16}$$

In accordance with (38) and the earlier remark, in connection with (48), concerning the meaning of $\underset{\sim}{I}_E$, the only contribution of $\underset{\sim}{J}_E$ to the magnetic field in V_S is proportional to the radial component of $\underset{\sim}{I}_E$. It follows from (B-16) that this contribution is zero up to terms of order less than a^2; thus, as promised, the contribution of $\underset{\sim}{J}_E$ to the magnetic field can be ignored in calculating the field to the first order in a.

APPENDIX C

SUMMARY OF FORMULAS FOR COMPUTING THE SPECTRA OF MAGNETIC FIELD GRADIENTS

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SUMMARY OF FORMULAS FOR COMPUTING THE SPECTRA OF MAGNETIC FIELD GRADIENTS

Formulas needed for calculating the magnetic field gradient spectrum appear in the main text. In fact, it is only necessary to apply the following equations, considered in the following order rather than the order in which they were originally introduced, to establish the necessary computational logic: (156), (157), (151), (160), (161), (159), (163), (170), (166), (169), (167), (178), (179), (175), (177), (135), (124).

The key parameters in the calculation are given by equations (136), reproduced here for convenience:

$$L_1 = \frac{\sigma\mu_o}{2} (\underset{\sim}{K} \cdot \underset{\sim}{B}_o)(K_x\Lambda_{1y}+K_z\Lambda_{2y}),$$

$$L_2 = -\frac{i\sigma\mu_o}{2K^2}[K_x(\underset{\sim}{\Lambda}_1 \cdot \underset{\sim}{K})+K_z(\underset{\sim}{\Lambda}_2 \cdot \underset{\sim}{K})](\underset{\sim}{K} \cdot \underset{\sim}{B}_o) - \frac{i\sigma\mu_o}{2}\left[\left(K_x\Lambda_{1y}+K_z\Lambda_{2y}\right)B_{oy} + (\underset{\sim}{K} \cdot \underset{\sim}{B}_o)\Lambda_{3y}\right],$$

$$L_3 = -\frac{\sigma\mu_o}{2K^2}\left\{(\underset{\sim}{\Lambda}_3 \cdot \underset{\sim}{K})(\underset{\sim}{B}_o \cdot \underset{\sim}{K})+[K_x(\underset{\sim}{\Lambda}_1 \cdot \underset{\sim}{K})+K_z(\underset{\sim}{\Lambda}_2 \cdot \underset{\sim}{K})]B_{oy}\right\} - \frac{\sigma\mu_o}{2}\Lambda_{3y}B_{oy},$$

$$L_4 = -\frac{i\sigma\mu_o}{2K^2}(\underset{\sim}{\Lambda}_3 \cdot \underset{\sim}{K})B_{oy}, \qquad (136)$$

$$M_1 = \frac{\sigma\mu_o}{25K}[(K_z\Gamma_{1x}-K_x\Gamma_{1z})K_x+(K_z\Gamma_{2x}-K_x\Gamma_{2z})K_z],$$

$$M_2 = i\frac{\sigma\mu_o}{25K}(-K_z\Gamma_{3x}+K_x\Gamma_{3z}).$$

For purposes of comparison and interpretation the various quantities, (136) can be expressed in terms of geometrical and physical parameters that were introduced in [1]. Each of those quantities is a function of two indices that are determined by the subscripts on the symbol for the magnetic field gradient spectrum. The indices of interest are the pairs 12, 13, 23; for these index pairs the following quantities appearing in (136) are defined:

$$
\begin{aligned}
&(\underset{\sim}{\Lambda}_1 \cdot \underset{\sim}{K})_{12} = 0, \ (\underset{\sim}{\Lambda}_1 \cdot \underset{\sim}{K})_{13} = K \sin(w-2\alpha), \ (\underset{\sim}{\Lambda}_1 \cdot \underset{\sim}{K})_{23} = 0, \\
&(\underset{\sim}{\Lambda}_2 \cdot \underset{\sim}{K})_{12} = 0, \ (\underset{\sim}{\Lambda}_2 \cdot \underset{\sim}{K})_{13} = K \cos(w-2\alpha), \ (\underset{\sim}{\Lambda}_2 \cdot \underset{\sim}{K})_{23} = 0, \\
&(\underset{\sim}{\Lambda}_3 \cdot \underset{\sim}{K})_{12} = K \cos(w-\alpha), \ (\underset{\sim}{\Lambda}_3 \cdot \underset{\sim}{K})_{13} = 0, \ (\underset{\sim}{\Lambda}_3 \cdot \underset{\sim}{K})_{23} = K \sin(w-\alpha), \\
&\Lambda_{1y12} = \cos\alpha, \ \Lambda_{1y13} = 0, \ \Lambda_{1y23} = -\sin\alpha, \\
&\Lambda_{2y12} = \sin\alpha, \ \Lambda_{2y13} = 0, \ \Lambda_{2y23} = \cos\alpha, \qquad\qquad \text{(C-1)} \\
&\Lambda_{3y12} = 0, \ \Lambda_{3y13} = 0, \ \Lambda_{3y23} = 0, \\
&\Gamma_{1x12} = 0, \ \Gamma_{1x13} = -B_o \sin\phi_D \cos 2\alpha, \ \Gamma_{1x23} = 0, \\
&\Gamma_{1z12} = -2B_o\cos\phi_D\cos\alpha, \ \Gamma_{1z13} = -B_o\sin\phi_D\sin 2\alpha, \ \Gamma_{1z23} = 2B_o\cos\phi_D\sin\alpha, \\
&\Gamma_{2x12} = B_o\cos\phi_D\cos\alpha, \ \Gamma_{2x13} = -B_o\sin\phi_D\sin 2\alpha, \ \Gamma_{2x23} = -B_o\cos\phi_D\sin\alpha, \\
&\Gamma_{2z12} = -B_o\cos\phi_D\sin\alpha, \ \Gamma_{2z13} = B_o\sin\phi_D\cos 2\alpha, \ \Gamma_{2z23} = -B_o\cos\phi_D\cos\alpha, \\
&\Gamma_{3x12} = -2B_o\sin\phi_D\sin\alpha, \ \Gamma_{3x13} = -B_o\cos\phi_D\cos 2\alpha, \ \Gamma_{3x23} = -2B_o\sin\phi_D\cos\alpha, \\
&\Gamma_{3z12} = 2B_o\sin\phi_D\cos\alpha, \ \Gamma_{3z13} = -B_o\cos\phi_D\sin 2\alpha, \ \Gamma_{3z23} = -2B_o\sin\phi_D\sin\alpha,
\end{aligned}
$$

where

$$B_o = \frac{2B_D}{1 + 3\cos^2\phi_D} .$$

Equations (C-1) used in (136) define all of the L_i and M_i corresponding to the subscript pairs 12, 13, 23. These are the parameters to be used when the sphere exists. In the absence of the sphere the M_i are set equal to zero, L_1 and L_3 are replaced by $\sigma\mu_o T_1$ and $\sigma\mu_o T_3$, and L_2 and L_4 are replaced by $i\sigma\mu_o T_2$ and $i\sigma\mu_o T_4$, where

$$T_{1\nu 1} = T_{1\nu 3} = T_{12\mu} = 0,\ T_{112} = -K^2 B_o \cos(w-\alpha)\cos w \cos \phi_D,$$

$$T_{132} = -K^2 B_o \sin(w-\alpha) \cos w \cos \phi_D,\ T_{221} = T_{223} = 0,$$

$$T_{211} = KB_o \cos^2(w-\alpha)\cos w \cos \phi_D,\ T_{212} = -KB_o \cos(w-\alpha)\sin \phi_D,$$

$$T_{213} = \tfrac{1}{2}KB_o \sin 2(w-\alpha)\cos w \cos \phi_D,\ T_{222} = -KB_o \cos w \cos \phi_D,$$

$$T_{231} = \tfrac{1}{2}KB_o \sin 2(w-\alpha)\cos w \cos \phi_D,\ T_{232} = -KB_o \sin(w-\alpha)\sin \phi_D,$$

$$T_{233} = KB_o \sin^2(w-\alpha)\cos w \cos \phi_D,\ T_{312} = T_{332} = 0,$$

$$T_{311} = -B_o \cos^2(w-\alpha)\sin \phi_D,\ T_{313} = -B_o \sin 2(w-\alpha)\sin \phi_D,$$

$$T_{321} = -B_o \cos(w-\alpha)\cos w \cos \phi_D,\ T_{322} = B_o \sin \phi_D,$$

$$T_{323} = -B_o \sin(w-\alpha)\cos w \cos \phi_D,$$

$$T_{331} = -\tfrac{1}{2}B_o \sin 2(w-\alpha) \sin \phi_D,\ T_{333} = -B_o \sin^2(w-\alpha)\sin \phi_D. \quad \text{(C-2)}$$

The quantities $L_{i\mu\nu}$, $M_{i\mu\nu}$ or, in the case of no sphere, the $T_{i\mu\nu}$ are used to define the quantities C_{o1} and C_{o2}, given by (161), the quantities C_{21} and C_{22} given by (167), the functions $F_o(y)$ and $F_1(y)$ given by (166), the functions $F_2(y)$ and $F_3(y)$ given by (169), the functions $G_0(y)$ and $G_1(y)$ given by (175), and the functions $G_2(y)$ and $G_3(y)$ given by (177). For convenience these equations are all repeated, as follows:

$$C_{o1} = \frac{1}{4}\left(\frac{1}{K^2} M_{1pq}M^*_{1rs} + M_{2pq}M^*_{2rs}\right)$$

$$C_{o2} = \frac{1}{4K}\left(M_{1pq}M^*_{2rs}+M_{2pq}M^*_{1rs}\right). \quad (161)$$

$$C_{21} = \frac{1}{2K}(L_1 + K^2 L_3),$$

$$C_{22} = \frac{1}{2}(L_2 + K^2 L_4). \qquad (167)$$

$$F_o(y) = K[(L_3 - \frac{C_{21}}{K}) \sinh Ky + KL_4 \cosh Ky + \frac{1}{2}(C_{21}+C_{22})ye^{Ky}$$

$$+ \frac{1}{2}(C_{21}-C_{22})ye^{-Ky}, \qquad (166)$$

$$F_1(y) = -\frac{1}{2}K(C_{21}-C_{22})e^{-Ky}$$

$$F_2(y) = \frac{K}{2}[L_3 + KL_4 - \frac{1}{K}C_{21} + (C_{21} + C_{22})y]e^{Ky} \qquad (169)$$

$$F_3(y) = \frac{K}{2}(C_{21} + C_{22})e^{Ky}.$$

$$G_o(y) = \frac{1}{K}(C_{21}+C_{22})(\frac{\sinh Ky}{K} - ye^{Ky}) + \frac{1}{2K}(C_{21}-C_{22})(\frac{\sinh Ky}{K} - ye^{-Ky})$$

$$-L_3 \frac{\sinh Ky}{K} - L_4 \cosh Ky \qquad (175)$$

$$G_1(y) = \frac{1}{2K}(C_{21}-C_{22})e^{-Ky}.$$

$$G_2(y) = \frac{1}{2}[\frac{C_{21}}{K^2} - \frac{L_3}{K} - L_4 - \frac{1}{K}(C_{21}+C_{22})y]e^{Ky}.$$

$$G_3(y) = -\frac{1}{2K}(C_{21}+C_{22})e^{Ky}. \qquad (177)$$

In addition, it is convenient to define

$$D = -\frac{1}{2}(\frac{M_1}{K} + M_2). \qquad (C-3)$$

For the case in which

$$I(\underset{\sim}{K}) = CK^{-p}$$

equation (156) for the magnetic field gradient spectrum can be written

$$\Phi_{\nu\mu;rs}(\omega,y) = \frac{2C}{\omega}\int_0^{\frac{\pi}{2}} K^{-(p+1)}\left[J^{(o)}_{\nu\mu;rs}(\alpha+\beta-\pi,K)+J^{(o)}_{\nu\mu;rs}(\alpha-\beta-\pi,K)\right]d\beta$$

$$+\frac{2C}{\omega}\int_0^{\frac{\pi}{2}} K^{-(p+1)}\left[J^{(1)}_{\nu\mu;rs}(\alpha+\beta-\pi,K)+J^{(1)}_{\nu\mu;rs}(\alpha-\beta-\pi,K)\right]d\beta, \tag{C-4}$$

where

$$J^{(o)}_{\nu\mu;rs}(w,K) = K^4\Big[G_{2\nu\mu}G^*_{2rs}f_o(-2K,y)-2G_{\nu\mu}G^*_{rs}f_o(o,y)$$

$$+(G_{2\nu\mu}G^*_{2rs} - G_{o\nu\mu}\,G^*_{ors})f_o(2K,y)$$

$$+ G_{o\nu\mu}G^*_{ors}\ f_o(2K,-\infty) - (G_{2\nu\mu}G^*_{3rs}+G_{3\nu\mu}G^*_{2rs})f_1(o,y)$$

$$+(G_{2\nu\mu}G^*_{3rs}+G_{3\nu\mu}G^*_{2rs} - G_{1\nu\mu}G^*_{ors} - G_{o\nu\mu}G^*_{1rs})f_1(2K,y)$$

$$+(G_{1\nu\mu}G^*_{ors} + G_{o\nu\mu}G^*_{1rs})f_1(2K,-\infty)$$

$$+(G_{3\nu\mu}G^*_{3rs} - G_{1\nu\mu}G^*_{1rs})f_2(2K,y) + G_{1\nu\mu}G^*_{1rs}\ f_2(2K,-\infty)\Big]$$

and

$$J^{(1)}_{\nu\mu;rs}(w,K) = \left[(C_{o1} + C_{o2})e^{2Ky} - (D^*_{\nu\mu}\ F^*_{2rs} + D_{rs}\ F_{2\nu\mu})e^{-Ky}\right]f_o(-2K,y)$$

$$+(D^*_{\nu\mu}F^*_{2rs} + D_{rs}\ F_{2\nu\mu})e^{-Ky}f_o(o,y)$$

$$+\left[(C_{o2} - C_{o1})e^{-2Ky} - (D_{\nu\mu}\ F^*_{ors} + D^*_{rs}\ F_{o\nu\mu})e^{-Ky}\right]f_o(2K,y)$$

$$+\left[(C_{o1} - C_{o2})e^{-2Ky} + (D_{\nu\mu}\ F^*_{ors} + D^*_{rs}\ F_{o\nu\mu})e^{-Ky}\right]f_o(2K,-\infty)$$

$$+\left[(D^*_{\nu\mu}\ F^*_{3rs} + D_{rs}\ F_{3\nu\mu})e^{-Ky}\right]f_1(-2K,y)$$

$$-\left[(D_{\nu\mu}\ F^{*}_{1rs} + D^{*}_{rs}\ F_{1\nu\mu})e^{-Ky}\right] f_1(2K,y)$$

$$+\left[(D_{\nu\mu}\ F^{*}_{1rs} + D^{*}_{rs}\ F_{1\nu\mu})e^{-Ky}\right] f_1(2K,-\infty), \qquad \text{(C-5)}$$

expanded in terms of the moments defined by (159), i.e., the functions

$$f_\lambda(K,y) = \int_y^0 x^\lambda\ e^{Kx} N^2(x)dx.$$

To save space, in (C-5) the y dependence of the functions $F_i(y)$ and $G_i(y)$ is not indicated. In (C-4) the integrand is separated into the two functions $J^{(o)}$ and $J^{(1)}$ because in the absence of the sphere $J^{(1)}$ vanishes leaving only $J^{(o)}$ suitably modified by the rule requiring the replacement of the quantities L_i with T_i and the omission of the M_i.

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